EFFICIENT TOOLS FOR QUANTUM METROLOGY WITH DECOHERENCE

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- Rafal Demkowicz-Dobrzanski, JK, Madalin Guta –
  ”*The elusive Heisenberg limit in quantum metrology*”, **Nat. Commun. 3, 1063 (2012)**.
- JK, Rafal Demkowicz-Dobrzanski –
  “*Efficient tools for quantum metrology with uncorrelated noise*”, **arXiv 1303.7271 (2013)**.
  (to be appear in New J. Phys.)

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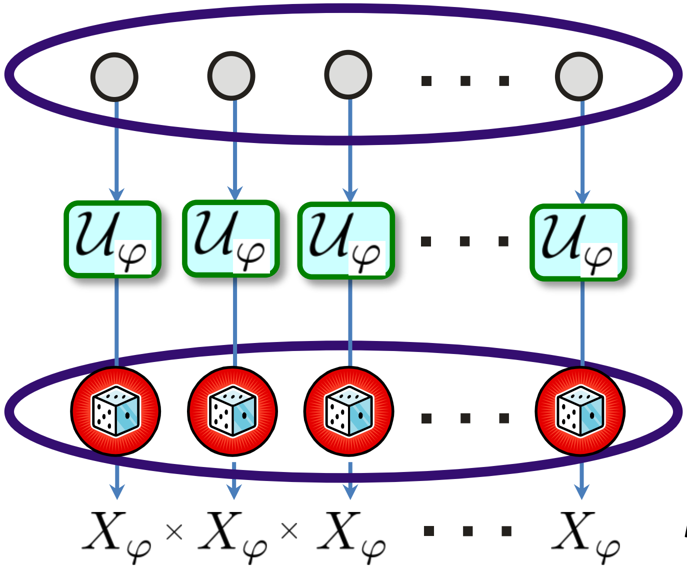
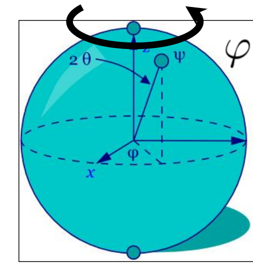
(Classical) Quantum Metrology

Atomic Spectroscopy: "Phase" Estimation



N **atoms in a <u>separable</u> state** $|\psi_{\rm in}^N\rangle = |\psi_{\rm in}^1\rangle^{\otimes N} = |+\rangle^{\otimes N}$

$$\hat{H}_N = \frac{\omega}{2}\sum_{i=1}^{N}\hat{\sigma}_z^{(i)}$$



$\varphi = \omega t$

unitary rotation $U_\varphi = \mathrm{e}^{\mathrm{i}\frac{\varphi}{2}\hat{\sigma}_z}$

output state $|\psi_{\rm out}^N\rangle = U_\varphi^{\otimes N}|\psi_{\rm in}^N\rangle = \left[\frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{\varphi}{2}}|0\rangle + \mathrm{e}^{\mathrm{i}\frac{\varphi}{2}}|1\rangle\right)\right]^{\otimes N}$

measurement – acc. to POVM $X_\varphi \sim p(x_i|\varphi) = \langle\psi_{\rm out}^1|\hat{M}_i^{(1)}|\psi_{\rm out}^1\rangle$

independent processes ⇒ $X_\varphi^N \sim p(X^{\times N}|\varphi) = p(X|\varphi)^N$

estimator

$$\tilde{\varphi}(X_1,\dots,X_N) \underset{N\to\infty}{\lesssim} \mathcal{N}\!\left(\varphi, \frac{1}{N\,F_{cl}[p(X|\varphi)]}\right) \underset{N\to\infty}{\lesssim} \mathcal{N}\!\left(\varphi, \frac{1}{N\,F_Q[|\psi_{\rm out}^1\rangle]}\right)$$

$$F_{cl}[p(X|\varphi)] = \int dx\,\frac{[\partial_\varphi p(x|\varphi)]^2}{p(x|\varphi)} \quad \leq \quad F_Q\left[|\psi_{\rm out}^1\rangle\right]$$

Classical Fisher Information

Quantum Fisher Information (strategy independent)

$$\Delta\tilde{\varphi} = \frac{1}{\sqrt{F_Q[|\psi_{\rm out}^1\rangle]}}\cdot\frac{1}{\sqrt{N}}$$

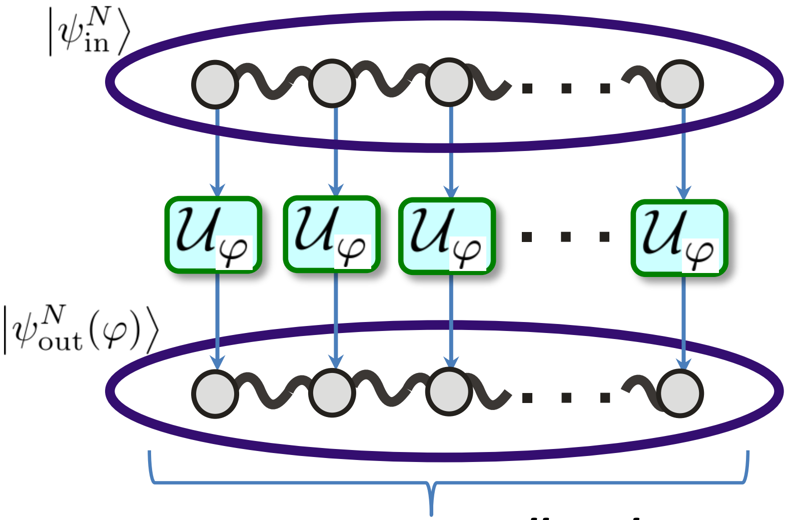
shot noise

saturable in the limit $N \to \infty$

- **As the *asymptotic N limit* is equivalent to <u>infinite number of repetitions</u> the ultimate precision is achievable in a <u>single experimental shot</u> despite the *locality* of *QFI*.**

(IDEAL) QUANTUM METROLOGY

ATOMIC SPECTROSCOPY: "PHASE" ESTIMATION



GHZ state

$$|\psi_{\mathrm{in}}^N\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right)$$

unitary rotation

$$U_\varphi = \mathrm{e}^{\mathrm{i}\varphi\hat{\sigma}_z}$$

output state

$$|\psi_{\mathrm{out}}^N(\varphi)\rangle = U_\varphi^{\otimes N}\,|\psi_{\mathrm{in}}^N\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{N\varphi}{2}}\,|0\rangle^{\otimes N} + \mathrm{e}^{\mathrm{i}\frac{N\varphi}{2}}\,|1\rangle^{\otimes N}\right)$$

measurement on all probes:

$$\left\{\hat{M}_i\right\}_{i=1}^{2^N} \text{ s.t. } p(i|\varphi) = \langle\psi_{\mathrm{out}}^N|\,\hat{M}_i\,|\psi_{\mathrm{out}}^N\rangle$$

estimator:

$$\tilde{\varphi}(i)$$

repeat the procedure $k \to \infty$ times !!!

$$\tilde{\varphi} \underset{k\to\infty}{\sim} \mathcal{N}\left(\varphi, \frac{1}{k\,F_Q\left[|\psi_{\mathrm{out}}^N(\varphi)\rangle\right]}\right)$$

$$F_Q\left[|\psi_{\mathrm{out}}^N(\varphi)\rangle\right] \sim N^2$$

$N \to \infty$ is not enough

atoms behaving as a "single object" with N times greater phase change generated.

"Real" resources are kN and in theory we require $k \to \infty$.

$$\Delta\tilde{\varphi} \underset{k\to\infty}{\sim} \frac{1}{\sqrt{k}N} \sim \frac{1}{N}$$

Heisenberg limit

- **But in real experiments there always exists a source of uncorrelated decoherence acting <u>independently</u> on each atom.**

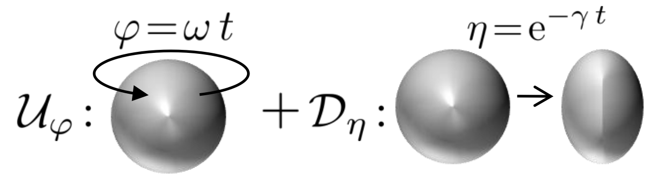
- **Such *decoherence* could "decorrelate" the atoms, so that we may attain the ultimate precision in the $N \to \infty$ limit with $k = 1$. But at the price of scaling …**

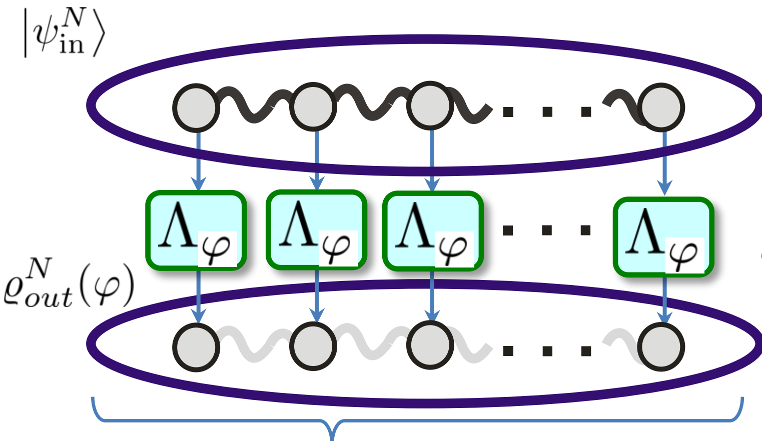
(Realistic) Quantum Metrology

Atomic Spectroscopy: "Phase" Estimation

*with **dephasing** noise added:*

$$\frac{\mathrm{d}\varrho^N}{\mathrm{d}t} = \mathrm{i}\frac{\omega}{2}\left[\varrho^N, \hat{H}_N\right] - \frac{\gamma}{2}\sum_{i=1}^{N}\left[\sigma_z^{(i)}\varrho^N\sigma_z^{(i)} - \varrho^N\right]$$





optimal pure state

distorted unitary rotation

mixed output state

<u>In principle</u> *need to optimize for particular model and* ***N***

$$\Lambda_\varphi\left[|\psi_{\mathrm{in}}^N\rangle\right] = \mathcal{D}_\eta\left[\mathcal{U}_\varphi\left[|\psi_{\mathrm{in}}^N\rangle\right]\right] = \mathcal{U}_\varphi\left[\mathcal{D}_\eta\left[|\psi_{\mathrm{in}}^N\rangle\right]\right]$$

$F_Q\left[\varrho_{out}^N(\varphi)\right]$ complexity of computation grows exponentially with *N*

measurement on all probes:

$$\left\{\hat{M}_i\right\}_{i=1}^{2^N} \text{ s.t. } p(i|\varphi) = \langle\psi_{\mathrm{out}}^N|\,\hat{M}_i\,|\psi_{\mathrm{out}}^N\rangle$$

estimator: $\tilde{\varphi}(i)$

$$\tilde{\varphi} \underset{N\to\infty}{\lesssim} \mathcal{N}\left(\varphi, \frac{c_{\mathrm{Q}}(\eta)}{N}\right)$$

$$\Delta\tilde{\varphi} \underset{N\to\infty}{\geq} \frac{\sqrt{1-\eta^2}}{\eta}\frac{1}{\sqrt{N}}$$

constant factor improvement over shot noise

achievable with k=1 and spin-squeezed states

Conclusions

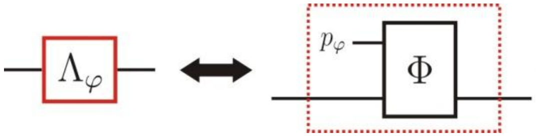
- **<u>Infinitesimal</u> uncorrelated disturbance forces asymptotic (classical) shot noise scaling.**
- **The bound then makes sense for a single shot (*k*=1).**
- **This occurs for <u>decoherence of a generic type</u>...**

The properties of the <u>single</u> use of a channel – Λ_φ – dictate the asymptotic ultimate scaling of precision.

EFFICIENT TOOLS FOR ~~DETERMINING~~ LOWER-BOUNDING $c_{\mathrm{Q}}(\eta)$

In order of their power and range of applicability:

- **Classical Simulation (CS) method**
  - Stems from the possibility to **simulate locally quantum chanels via classical probabilistic mixtures**:



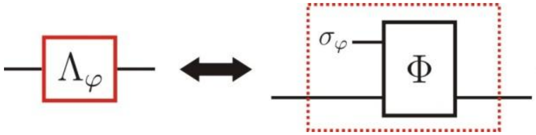
$+\,O(\delta\varphi^2)$

$$\Lambda_\varphi[\varrho] = \Phi[\varrho \otimes p_\varphi] + O(\delta\varphi^2) = \textstyle\sum_i p_i(\varphi)\, \Pi_i[\varrho] + O(\delta\varphi^2)$$

$$p_\varphi = \textstyle\sum_i p_i(\varphi)\, |e_i\rangle\langle e_i|$$

  - Optimal simulation corresponds to a **simple, intuitive, geometric representation**.
  - Proves that ***almost all* (including full rank)** channels **asymptotically scale classically**.
  - Allows to **straightforwardly derive bounds** (e.g. dephasing channel considered).

- **Quantum Simulation (QS) method**
  - Generalizes the concept of local classical simulation, so that the parameter-dependent state does not need to be diagonal:



$+\,O(\delta\varphi^2)$

$$\Lambda_\varphi[\varrho] = \Phi[\varrho \otimes \sigma_\varphi] + O(\delta\varphi^2)$$

  - Proves **asymptotic shot noise** also for **a wider class of channels** (e.g. optical interferometer with loss).

- **Channel Extension (CE) method**
  - Algebraic method that applies to **even wider class than quantum** (and classically) **simulable** channels (e.g. with noise due to spontaneous emission), and provides the tightest lower bounds on $c_{\mathrm{Q}}(\eta)$.
  - Can be **efficiently performed numerically** by means of **Semi-Definite Programming**.
  - Its **numerical form** can be improved and applied to the **finite-N regime**.

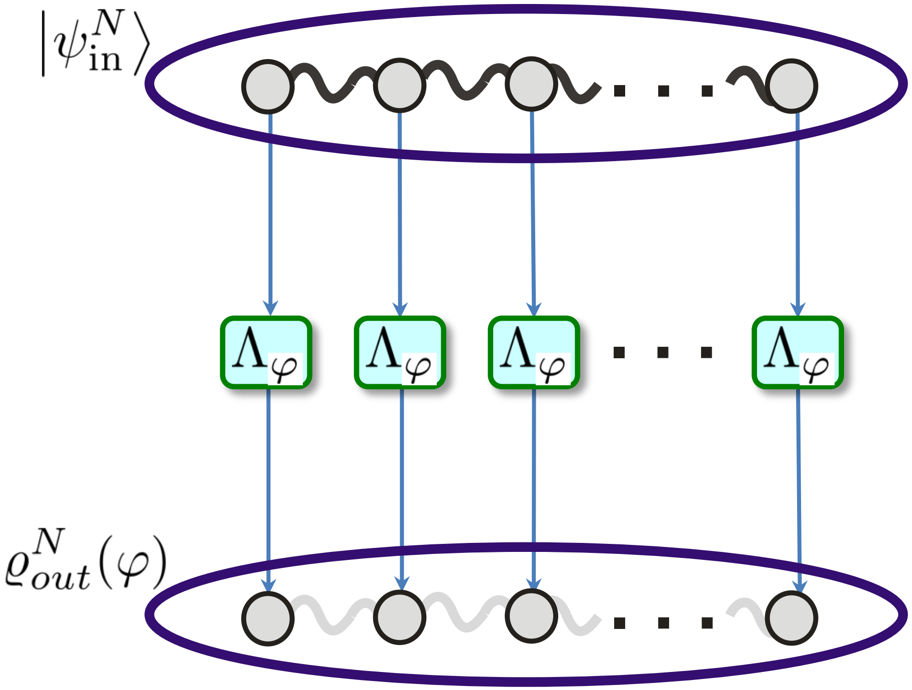
[JK, R. Demkowicz-Dobrzanski – **arXiv 1303.7271(2013)**]

    - the **finite-N CE method** has been successfully applied to prove the possibility of ***$1/N^{5/6}$*** (**beating shot noise!**) asymptotic scaling with noise being the ***transversal dephasing***.

[R. Chaves, J. B. Brask, M. Markiewicz, JK, A. Acin – **arXiv 1212.3286 (2013)**]

(see the poster of Marcin Markiewicz)

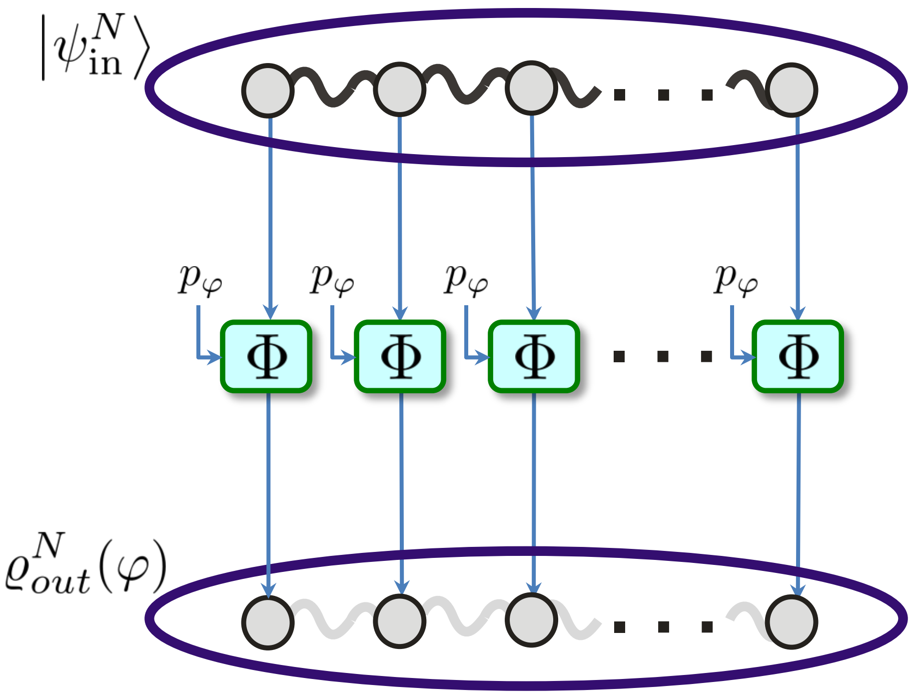
CLASSICAL/QUANTUM SIMULATION OF A CHANNEL



as a Markov chain:

$$\varphi \rightarrow \Lambda_\varphi^{\otimes N}\left[|\psi_{\rm in}^N\rangle\right] \rightarrow \tilde{\varphi} \quad \Longrightarrow \quad \Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F_Q[\varrho_{out}^N(\varphi)]}}$$

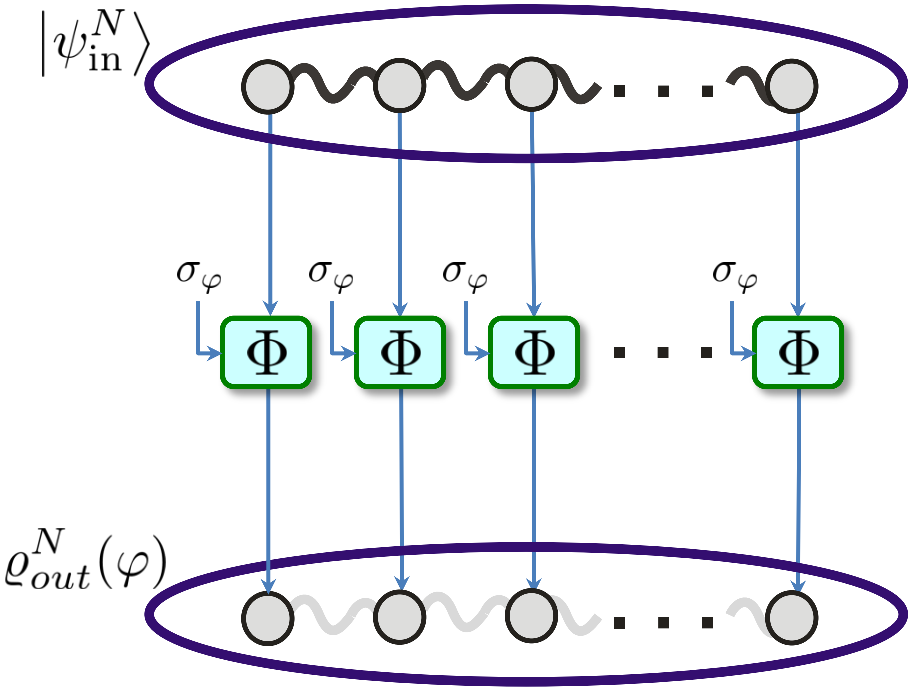
Classical/Quantum Simulation of a channel



as a Markov chain:

$$\varphi \to p_\varphi \to \Phi[p_\varphi \otimes \bullet]^{\otimes N}\big(|\psi_{\rm in}^N\rangle\big) \to \tilde{\varphi} \quad \Longrightarrow \quad \Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F_Q[\varrho_{out}^N(\varphi)]}}$$

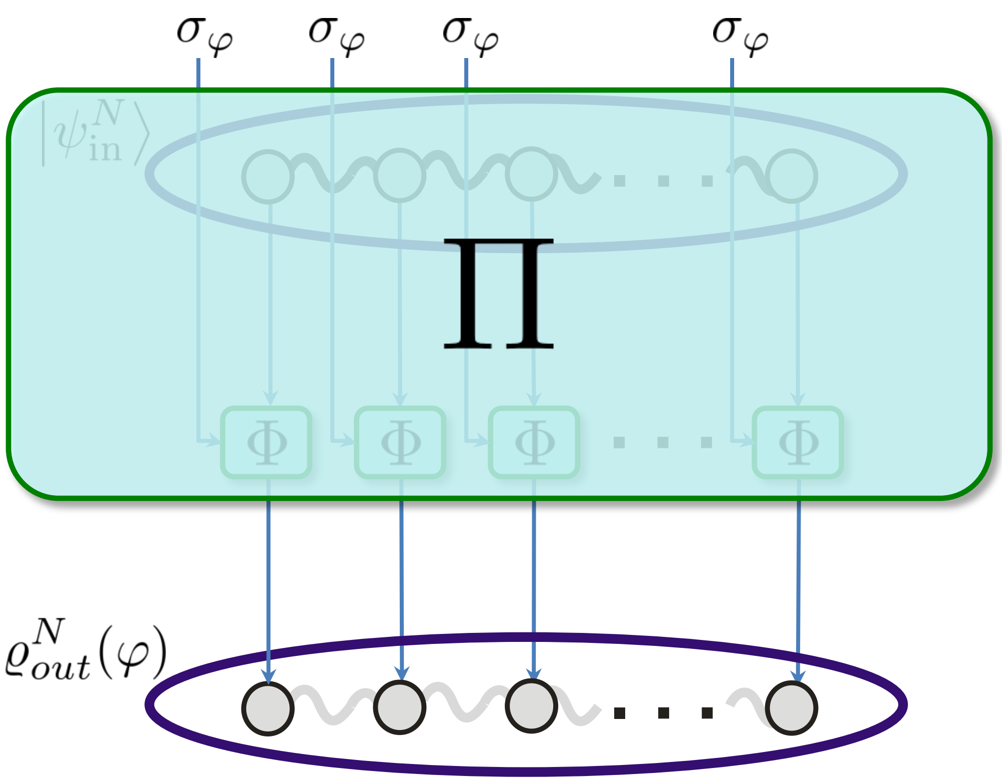
CLASSICAL/QUANTUM SIMULATION OF A CHANNEL



as a Markov chain:

$$\varphi \rightarrow \sigma_\varphi \rightarrow \Phi[\sigma_\varphi \otimes \bullet]^{\otimes N}\left(|\psi_{\text{in}}^N\rangle\right) \rightarrow \tilde{\varphi} \quad \Longrightarrow \quad \Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F_Q[\varrho_{out}^N(\varphi)]}}$$

CLASSICAL/QUANTUM SIMULATION OF A CHANNEL



as a Markov chain:

$$\varphi \to \sigma_\varphi \to \sigma_\varphi^{\otimes N} \to \Phi^{\otimes N}[\bullet \otimes |\psi_{\rm in}^N\rangle](\sigma_\varphi^{\otimes N}) = \Pi[\sigma_\varphi^{\otimes N}] \to \tilde{\varphi}$$

$$|\wedge$$

$$\varphi \to \sigma_\varphi \to \sigma_\varphi^{\otimes N} \to \tilde{\varphi}$$

shot noise scaling !!!

$$\Rightarrow \quad \Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F_Q[\varrho_{out}^N(\varphi)]}} \geq \frac{1}{\sqrt{F_Q[\sigma_\varphi]}} \frac{1}{\sqrt{N}}$$

THE "WORST" CLASSICAL SIMULATION

The set of quantum channels (CPTP maps) is *convex*

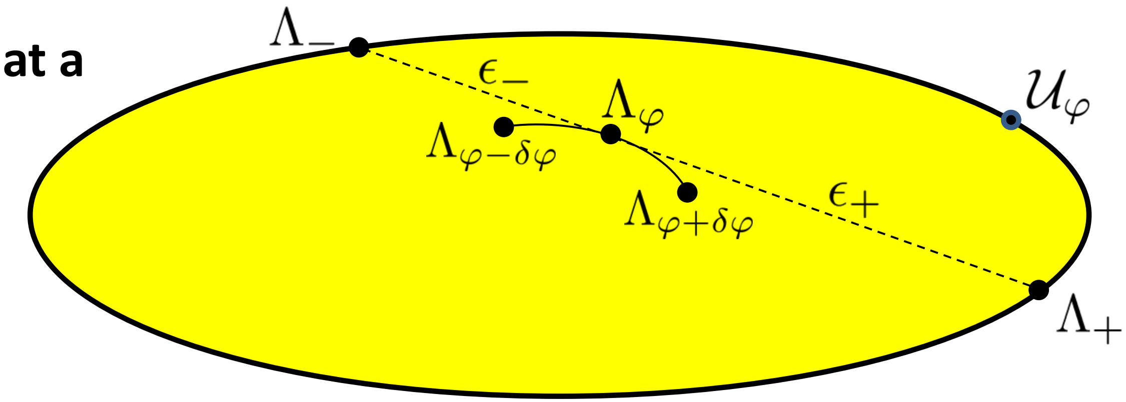
$$\Lambda: \ \varrho_{in} \in B(\mathcal{H}_{d_{in}}) \longrightarrow \varrho_{out} \in B(\mathcal{H}_{d_{out}})$$

***Locality*:**

Quantum Fisher Information at a given φ: $F_Q\left[\Lambda_\varphi^{\otimes N}\left[|\psi_{in}^N\rangle\right]\right]$

depends only on

$$\Lambda_\varphi \qquad \partial_\varphi \Lambda_\varphi$$



It is enough to analize „local classical simulation'':

$$\Lambda_\varphi[\varrho] = \Phi[\varrho \otimes p_\varphi] + O(\delta\varphi^2) = \sum_i p_i(\varphi)\, \Lambda_i[\varrho] + O(\delta\varphi^2)$$

The *„worst" local classical simulation*:

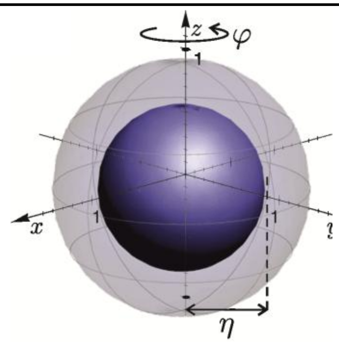
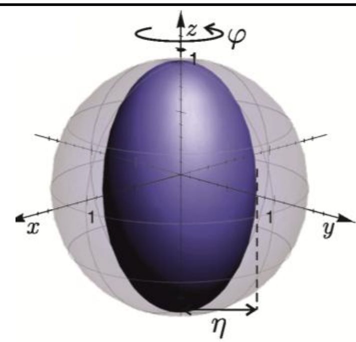
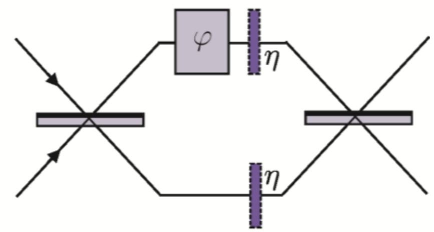
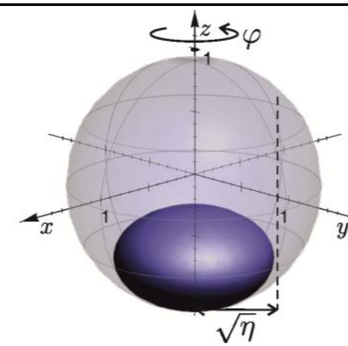
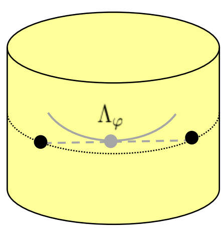
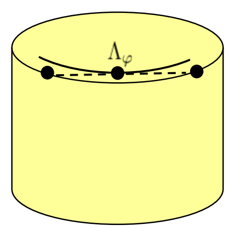
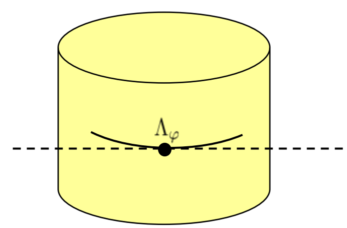
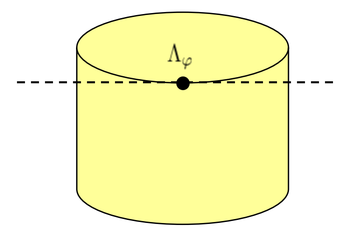
$$\Lambda_\varphi = p_+(\varphi)\Lambda_+ + p_-(\varphi)\Lambda_- + O(d\varphi^2) \qquad \Lambda_\pm = \Lambda_\varphi \pm \frac{d\Lambda_\varphi}{d\varphi}\epsilon_\pm$$

$$F_Q \leq F_Q^{\mathrm{CS}} = N\, F_{\mathrm{cl}}[p_\pm(\varphi)] = \frac{N}{\epsilon_+ \epsilon_-}$$

$$c_{\mathrm{Q}} = \epsilon_+\epsilon_-, \ \ \Delta\tilde{\varphi} \geq \sqrt{\frac{\epsilon_+\epsilon_-}{N}}$$

Does not work for φ -extremal channels, e.g *unitaries* $\mathcal{U}_\varphi$.

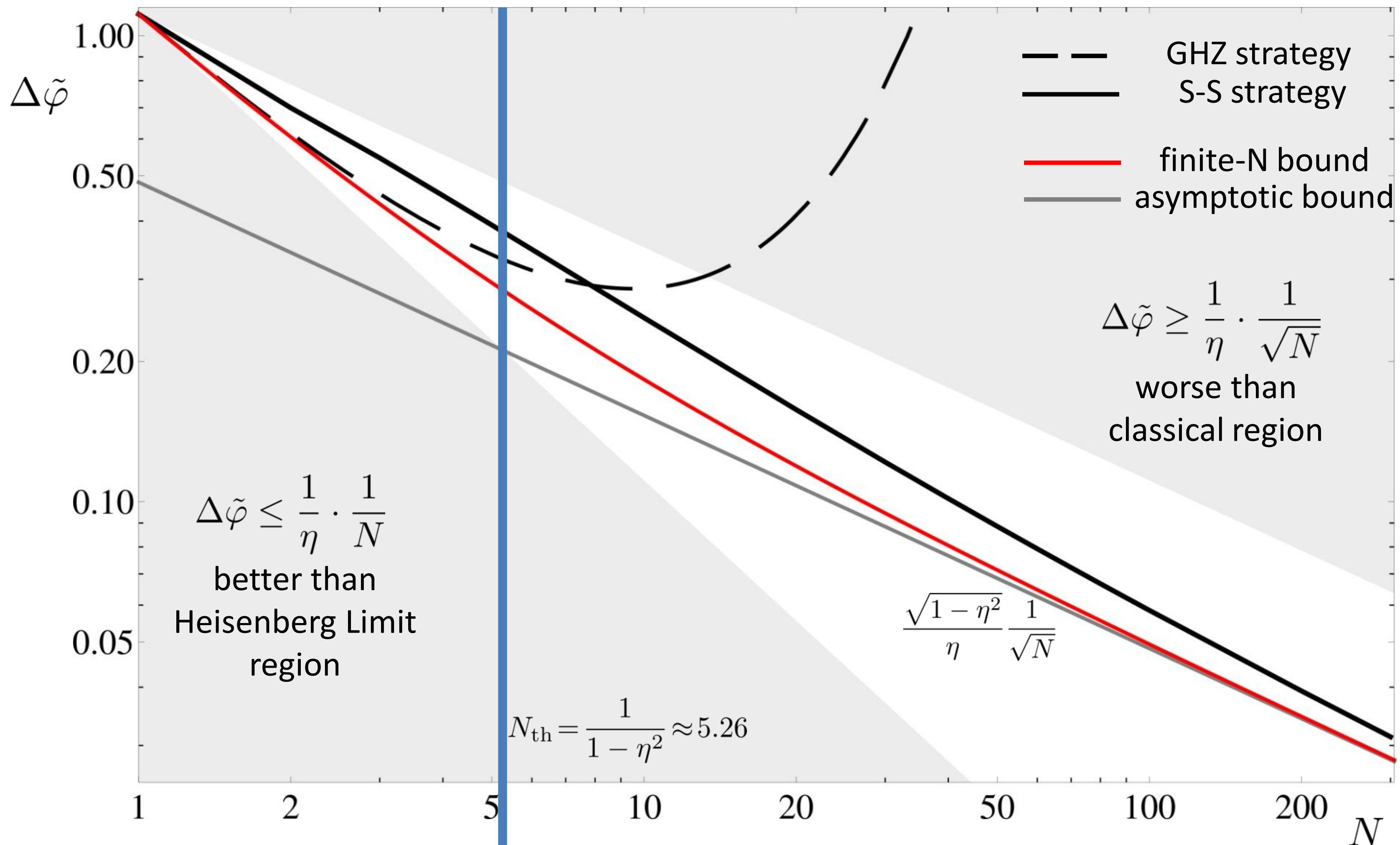
Gallery of decoherence models

	Depolarization	Dephasing	Lossy interferometer	Spontaneous emission
				
				
	inside the set of quantum channels **full rank**	on the boundary, non-extremal, **not φ-extremal**	on the boundary, non-extremal, **but φ-extremal**	on the boundary, extremal
CS	$\sqrt{\frac{1+3\eta}{4\eta}\cdot\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$	$\frac{\sqrt{1-\eta^2}}{\eta}\frac{1}{\sqrt{N}}$	N/A	N/A
QS	$\sqrt{\frac{1+2\eta}{2\eta}\cdot\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$	$\frac{\sqrt{1-\eta^2}}{\eta}\frac{1}{\sqrt{N}}$	$\sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$	N/A
CE	$\sqrt{\frac{1+2\eta}{2\eta}\cdot\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$	$\frac{\sqrt{1-\eta^2}}{\eta}\frac{1}{\sqrt{N}}$	$\sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$	$\frac{1}{2}\sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$

$$c_Q^{CE} \geq c_Q^{QS} \geq c_Q^{CS} \quad \Rightarrow \quad \Delta\tilde{\varphi} \geq \Delta\tilde{\varphi}_{CE} \geq \Delta\tilde{\varphi}_{QS} \geq \Delta\tilde{\varphi}_{CS}$$

Consequences on Realistic Scenarios

Atomic Spectroscpopy with Dephasing ($\eta = 0.9$) – "phase estimation" (fixed time t in $\omega \cdot t$)



input correlations dominated region

worth investing in GHZ states

e.g. N=3

[D. Leibfried et al, **Science**, 304 (2004)]

uncorrelated decoherence dominated region

worth investing in S-S states

e.g. N=10^5 !!!!!!!

[R. J. Sewell et al, **PRL** 109, 253605 (2012)]

CONCLUSIONS

- **Classically**, for **separable** input states, the ultimate precision is bound to **shot noise scaling** $\boldsymbol{1/\sqrt{N}}$, which can be attained in a single experimental shot ($\boldsymbol{k=1}$).
- For **lossless** unitary evolution highly **entangled** input states (*GHZ, N00N*) allow for ultimate precision that follows the **Heisenberg scaling** $\boldsymbol{1/N}$, but attaining this limit may in principle require <u>infinite repetitions of the experiment</u> ($\boldsymbol{k}\to\infty$).
- The consequences of the **dehorence** acting **independently** on each particle:
  - The **Heisenberg scaling** is lost and only a **constant factor quantum enhancement** over classical estimation strategies is allowed.
  - The **optimal input states** in the $N\to\infty$ limit are **of a simpler form** (*spin-squeezed atomic, squeezed light states*) and achieve the ultimate precision in a single shot ($\boldsymbol{k=1}$).
  - However, finding the **optimal form** of those states is **still an issue**. Classical scaling suggests local correlations → MPS states – (yesterday's talk by Marcin Jarzyna).
- We have formulated methods: **Classical Simulation**, **Quantum Simulation** and **Channel Extension**; that may be applied to <u>prove this behaviour</u> and efficiently <u>lower-bound the constant factor</u> of the quantum asymptotic enhancement for a generic channel.
- The **CE** method may also be applied numerically for **finite** N as a **semi-definite program**.
- The geometrical **CS** method proves the $\boldsymbol{c_Q/\sqrt{N}}$ for all **full-rank channels** and more.
  - Yet, by using a *cunning trick* we managed to find a channel that, **despite being full-rank,** achieves the ultimate $\boldsymbol{1/N^{5/6}}$ asymptotic scaling – the ***transversal dephasing***.

(see the poster of Marcin Markiewicz)