(Im-)Proving Landauer's Principle

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`arXiv:1306.xxxx, Landauer's Principle`
`arXiv:1304.0036, Lower bound on relative entropy (and applications)`

Landauer's Principle – a common formulation

Suppose a computer "erases" 1 bit of information.

Then: The amount of "heat" "dissipated" into the environment is at least $k_B T \log 2$:

$$\Delta Q \; \geq \; k_B T \log 2 \; ,$$

where $T =$ temperature of the environment of the computer.

$$\beta \Delta Q \; \geq \; \Delta S \qquad \text{"Landauer bound"}$$

where $k_B \; \equiv \; 1 \, , \;\; \beta \; \equiv \; 1/T$

Why "erasure"? E.g. to re-initialize error correcting mechanism.

Existing derivations of LP

- **based on 2^{nd} Law of Thermodyn:** e.g. Landauer '61, ...
  $\rightarrow$ mix-up of notions (cf. Earman/Norton, Bennett, ...)
- **in specific models:** e.g. 1-particle gas in box
  $\rightarrow$ need to accept thermodyn formalism (e.g. "quasistatic")
- **recently: (more) microscopic**
  - Shizume (1995): *effective* dissipative force (Fokker-Planck)
  - Piechocinska (2000): *Jarzynski equality*
    – assumes: final product state $\rho_S \otimes \rho_R \overset{U}{\mapsto} \rho'_S \otimes \rho'_R$
    – assumes: ρ'_S pure
    – assumes: ρ'_R diagonal in energy eigenbasis $\rightarrow$ *quantum?*
  - Sagawa/Ueda (2009): need system Hamiltonian H_S, ...
- **claimed "violations" of LP:**
  $\rightarrow$ Nieuwenhuizen '01, Lutz '11, Orlov '12, ...

our work: rigorous and minimal formulation & proof of LP

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1. Formulation & Proof
   - minimal setup
   - LP equality: $\beta\Delta Q \;=\; \Delta S + I(S' : R') + D(\rho_R' \| \rho_R)$
2. Finite-size effects
   - entropy inequalities
   - finite-size effects: $\beta\Delta Q \;\geq\; \Delta S + \frac{(\Delta S)^2}{\log^2 d}$, ...
3. Sharpness of $\beta\Delta Q \geq \Delta S$

Minimal assumptions for LP

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(1) system S, reservoir R: $\mathcal{H}_{SR} = \mathcal{H}_S \otimes \mathcal{H}_R$

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(2) initial state uncorrelated: $\rho_{SR} = \rho_S \otimes \rho_R$

"Counterexample": perfect classical correlations

Suppose: $\rho_{SR} = \sum_i p_i |i\rangle_S\langle i| \otimes |i\rangle_R\langle i|$.

Let: $U: |i\rangle_S|i\rangle_R \mapsto |0\rangle_S|i\rangle_R$.

Then: $U\rho_{SR}U^\dagger = |0\rangle_S\langle 0| \otimes \sum_i p_i |i\rangle_R\langle i|$

$\rightarrow$ $\rho'_R = \rho_R$

$\rightarrow$ no heat change

$\rightarrow$ LP "violated"

Minimal assumptions for LP

(1) system S, reservoir R: $\mathcal{H}_{SR} = \mathcal{H}_S \otimes \mathcal{H}_R$
(2) initial state uncorrelated: $\rho_{SR} = \rho_S \otimes \rho_R$
(3) $\rho_R = \frac{e^{-\beta H}}{\mathrm{tr}[e^{-\beta H}]}$ (R-Hamiltonian H, R-temperature $T \equiv 1/\beta$)

- parameter T or β occurring in LP
- "stable" states at ambient temperature T
- "freely available"

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(4) unitary evolution: $\rho'_{SR} = U \rho_{SR} U^\dagger$

- microscopic laws of nature
- avoid obscuring entropy sinks

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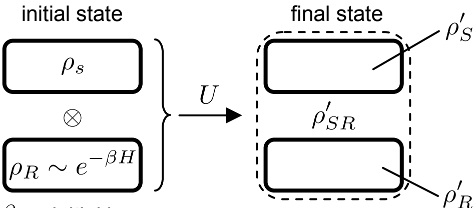
(4) unitary evolution: $\rho'_{SR} = U\rho_{SR}U^\dagger$

- no Hamiltonian for S
- no temperature for S
- ρ'_{SR} may be correlated
- ρ'_S need not be pure
- entropy of S may decrease or increase
- quantum ($[H, \rho'_R] \neq 0$) and classical

Minimal setup for LP



- $\Delta S \ := \ S(\rho_S) \ - \ S(\rho'_S)$
- $\Delta \ := \ S(\rho'_R) \ - \ S(\rho_R)$
- $\Delta Q \ := \ \mathrm{Tr}\,[H\rho'_R] \ - \ \mathrm{Tr}\,[H\rho_R]$

[von Neumann entropy: $S(\rho) \ := \ -\mathrm{Tr}\,[\rho \log \rho] \ = \ -\sum_i p_i \log p_i$

$\rightarrow$ *averaged* quantities; also: $\Delta Q =$ *averaged* heat flow]

Proof of LP

1. "Second Law Lemma": $\Delta = \Delta S + I(S' : R') \geq \Delta S$

Proof

$$\begin{aligned}
\Delta - \Delta S &= S(\rho_R') - S(\rho_R) + S(\rho_S') - S(\rho_S) \\
&= S(\rho_S') + S(\rho_R') - S(\rho_{SR}) \\
&= S(\rho_S') + S(\rho_R') - S(\rho_{SR}') \\
&= I(S' : R') \geq 0 \,.
\end{aligned}$$

– $I(A : B) \geq 0$ mutual information

– no thermal state assumption

Proof of LP

1. "Second Law Lemma": $\Delta \;=\; \Delta S + I(S' : R') \;\geq\; \Delta S$
2. R-entropy vs. heat: $\beta \Delta Q \;=\; \Delta + D(\rho'_R \| \rho_R)$

Proof

$$\begin{aligned}\Delta &= S(\rho'_R) - S(\rho_R)\\ &= \mathrm{tr}\Big[-\rho'_R \log \rho'_R \;+\; \rho_R \log \frac{e^{-\beta H}}{\mathrm{Tr}\,[e^{-\beta H}]}\Big]\\ &= \mathrm{tr}\Big[\beta H(\rho'_R - \rho_R)\Big] + \mathrm{tr}\Big[\rho'_R \log \frac{e^{-\beta H}}{\mathrm{Tr}\,[e^{-\beta H}]} - \rho'_R \log \rho'_R\Big]\\ &= \;\beta \Delta Q \;-\; D(\rho'_R \| \rho_R)\;.\end{aligned}$$

recall: $D(\sigma \| \rho) \;:=\; \mathrm{Tr}\,[\sigma \log \sigma] - \mathrm{Tr}\,[\sigma \log \rho] \;\geq\; 0$

("relative entropy")

Main result I: Equality form of LP

Theorem: Equality form of Landauer's Principle

Let $\rho_{SR} = \rho_S \otimes \rho_R$ be a product state,

where $\rho_R = e^{-\beta H}/\mathrm{Tr}\left[e^{-\beta H}\right]$ is thermal state of Hamiltonian H at inverse temperature β.

Assume $\rho'_{SR} := U\rho_{SR}U^{\dagger}$ with a unitary evolution U.

Then:

$$\begin{aligned} \beta\Delta Q &= \Delta S + I(S' : R') + D(\rho'_R \| \rho_R) \\ &\geq \Delta S\,. \end{aligned}$$

Equality cases in Landauer's bound: $\beta\Delta Q = \Delta S$

Equality form of LP: $\beta\Delta Q \;=\; \Delta S \,+\, I(S':R') \,+\, D(\rho'_R\|\rho_R)$

- $D(\rho'_R\|\rho_R) = 0 \;\;\Rightarrow\;\; \rho'_R \;=\; \rho_R$
- $I(S':R') = 0 \;\;\Rightarrow\;\; \rho'_{SR} \;=\; \rho'_S \otimes \rho_R \;\;=\;\; U\,(\rho_S \otimes \rho_R)\,U^\dagger$

Thus: $\beta\Delta Q = \Delta S \;\;\Leftrightarrow\;\; \rho'_S = V\rho_S V^\dagger$ and $\rho'_R = \rho_R$

$\Leftrightarrow\;\; \Delta S \;=\; \Delta Q \;=\; 0$

$\Leftrightarrow$ basically nothing happens

Next: explicit improvements of Landauer's bound:

$\rightarrow$ need: finite size $d = \dim(R) < \infty$

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Finite-size effects

Reservoir: $\dim(\mathcal{H}_R) = d < \infty$:

- e.g. when error-correcting mechanism small
- e.g. when short interaction time $S - R$: effectively small d

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Finite-size effects

Reservoir: $\dim(\mathcal{H}_R) = d < \infty$:

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Idea: $\beta\Delta Q \;\geq\; \Delta S + D(\rho'_R \| \rho_R)$

$\Delta S > 0$:

$$\Rightarrow \quad 0 < \Delta = S(\rho'_R) - S(\rho_R)$$
$$\Rightarrow \quad \rho'_R \neq \rho_R$$
$$\Rightarrow \quad D(\rho'_R \| \rho_R) > 0\,.$$

$\rightarrow$ new entropy inequality: $D(\rho'_R \| \rho_R) \geq M(\Delta, d)$

$\Delta S < 0$: $$D(\rho'_R \| \rho_R) \;\geq\; \frac{(\beta\Delta Q)^2}{2\max_T C(T)} \;\geq\; \frac{(\beta\Delta Q)^2}{2N(d)}$$

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Relative entropy vs. entropy difference

Theorem

Let σ, ρ on $\mathbb{C}^d$. Define $\Delta := S(\sigma) - S(\rho)$. **Then:**

$$D(\sigma\|\rho) \;\geq\; M(\Delta, d) \;\geq\; \frac{\Delta^2}{2N} + \frac{\Delta^3}{6N^2} \;\geq\; 0\,,$$

where

$$M(\Delta, d) := \min_{s,r} \left\{ D_2(s\|r) \,\middle|\, H_2(s) - H_2(r) + (s-r)\log(d-1) = \Delta \right\}$$

and $\quad N = \dfrac{1}{4}\log^2(d-1) + 1 \qquad$ or $\qquad N = \log^2 d\,.$

- $M(\Delta, d)$: tight, effectively computable, strictly convex
- $N(d) := \max_{0<r<1/2} r(1-r)\left(\log \frac{1-r}{r}(d-1)\right)^2$: cubic Taylor

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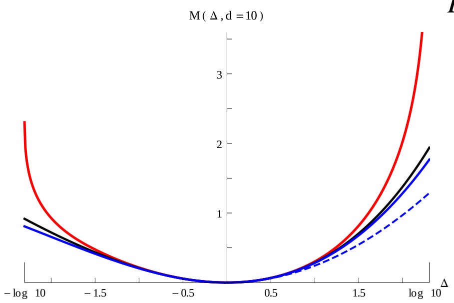
Relative entropy vs. entropy difference



graph for
$d = 10 = \dim(R)$

$$
\begin{aligned}
D(\rho'_R \| \rho_R) \;&\geq\; M(\Delta, d) \\
&\geq\; \frac{\Delta^2}{2N(d)} + \frac{\Delta^3}{6N(d)^2} \\
&\geq\; \frac{\Delta^2}{\frac{1}{2}\log^2(d-1) + 2} \\
&\geq\; 0\,.
\end{aligned}
$$

(better than Pinsker + Fannes–Audenaert)

Relative entropy vs. entropy difference

Proof

Let $\Delta = S(\sigma) - S(\rho)$ be given:

1. $D(\sigma\|\rho) = \mathrm{Tr}\left[(-\log\rho)\sigma\right] - S(\sigma)$ for fixed ρ, fixed $S(\sigma)$
   $\Rightarrow$ same exponential family: $\sigma = \rho^\gamma / \mathrm{Tr}\left[\rho^\gamma\right]$
2. Lagrange multipliers $\Rightarrow \rho, \sigma$ have at most 2 distinct EVs$\neq 0$
3. discrete optimization $\Rightarrow$ 1 large EV, $(d-1)$ small EVs

$$\sigma = \mathrm{diag}\left(1-s, \frac{s}{d-1}, \ldots, \frac{s}{d-1}\right)$$
$$\rho = \mathrm{diag}\left(1-r, \frac{r}{d-1}, \ldots, \frac{r}{d-1}\right)$$

classical states (commuting)

$$\Rightarrow M(\Delta, d) = \inf_{0\leq s,r\leq 1}\left\{D_2(s\|r) \,\middle|\, H_2(s) - H_2(r) + (s-r)\log(d-1) = \Delta\right\}$$

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Improved Landauer bound for $\Delta S \geq 0$

$$\beta\Delta Q \;\geq\; \Delta S + D(\rho_R' \| \rho_R) \;\geq\; \Delta S + M(\Delta S, d)$$
$$\geq\; \Delta S + \frac{(\Delta S)^2}{\frac{1}{2}\log^2(d-1) + 2}\,.$$

Example: Erasure of 1 bit of information

- 2-qubit reservoir: $> 34\%$ more heat dissipation
- 5-qubit reservoir: $> 12\%$ more heat dissipation

- bound tight
- denominator $\sim \log^2 d \sim (\#particles)^2$
- $\log d$ $=$ effective reservoir D.O.F.'s in *short* interaction

Finite-size effects for $\beta\Delta Q \leq 0$

$$
\begin{aligned}
D(\rho'_R \| \rho_R) &\geq D(\rho'_{R,th} \| \rho_R) = \beta\Delta Q - \left[S(\rho'_{R,th}) - S(\rho_R)\right] \\
&= \ldots \geq \\
&\geq \int_{E_R}^{E_R+\Delta Q} \int_{E_R}^{E} \frac{\beta^2}{C_H(E')}\, dE'\, dE \\
&\geq \frac{(\beta\Delta Q)^2}{2N(d)}
\end{aligned}
$$

[next slide: $C_H(E') \leq N(d) \leq \frac{1}{4}\log^2(d-1)+1$]

$$\Rightarrow \quad \beta\Delta Q \geq \Delta S + \frac{(\beta\Delta Q)^2}{2N(d)}$$

$$\Rightarrow \quad \beta\Delta Q \geq \Delta S + \left[N(d) - \Delta S + \sqrt{N(d)^2 - 2N(d)\Delta S}\right]$$

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Maximum heat capacity in $d < \infty$ dimensions

$$C_H(\beta) = \frac{d}{dT}\mathrm{tr}\left[H\frac{e^{-H/T}}{\mathrm{tr}\left[e^{-H/T}\right]}\right]\Bigg|_{T=\frac{1}{\beta}} = \mathrm{var}_{\rho_\beta}(\beta H) = \mathrm{var}_{\rho_\beta}(\log\rho_\beta)$$

$$= \mathrm{Tr}\left[\rho_\beta\left(\log\rho_\beta + S(\rho_\beta)\mathbb{1}\right)^2\right]$$

Theorem: For any state ρ on $\mathbb{C}^d$:

$$\mathrm{var}_\rho(\log\rho) \leq N(d) := \max_{0\leq r\leq 1/2} r(1-r)\left(\log\frac{1-r}{r}(d-1)\right)^2$$

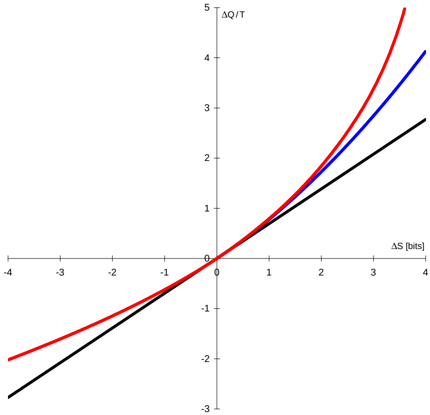
$$< \frac{1}{4}\log^2(d-1) + 1 \lesssim n^2 \quad \text{“\textit{superextensive}”}$$

attained for:

$$\rho = \mathrm{diag}\left(1-r, \frac{r}{d-1}, \ldots, \frac{r}{d-1}\right)$$

$$H = \mathrm{diag}(-1, 0, \ldots, 0)$$

Main result II: Finite-size improvements of LP



$d = 16 = \dim(R)$ (4-qubit reservoir)

- Landauer's bound:
  $\beta\Delta Q \geq \Delta S$
- tight bound for $\Delta S \geq 0$:
  $\beta\Delta Q \geq \Delta S + M(\Delta S, d)$
- for $\Delta S \leq 0$ (not tight):
  $\beta\Delta Q \geq \Delta S + \left[N_d - \Delta S - \sqrt{N_d^2 - 2N_d\Delta S}\right]$
- quadratic bound $\Delta S \geq 0$:
  $\beta\Delta Q \geq \Delta S + \frac{2(\Delta S)^2}{\log^2(d-1)+4}$

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Extension: Processes with memory

Let: $\rho_{SM} \otimes \rho_R \quad \mapsto \quad \rho'_{SMR} = U(\rho_{SM} \otimes \rho_R)U^\dagger$

Example: perfect classical correlations

$$\rho_{SM} = \sum_i p_i \, |i\rangle_S\langle i| \otimes |i\rangle_M\langle i| \quad \overset{U_{SM}}{\mapsto} \quad |0\rangle_S\langle 0| \otimes \sum_i p_i |i\rangle_M\langle i|$$

whereas $\rho'_R = \rho_R$, i.e. $\Delta = \Delta Q = 0$.

Result: If $\rho'_M = \rho_M$ [or $S(\rho'_M) \leq S(\rho_M)$], then:

1. **"2nd Law":** $\Delta \geq \Delta S_{cond} := S(S|M) - S(S'|M')$
2. **LP:** $\beta\Delta Q \geq \Delta S_{cond} + I(S'M' : R') + D(\rho'_R \| \rho_R) \geq \Delta S_{cond}$
   $\rightarrow$ **proofs** similar as before
3. **finite-size corrections** with ΔS_{cond} rather than ΔS

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Achievability of Landauer's bound

- How sharp can $\beta\Delta Q \geq \Delta S$ be ?
- Given ρ_S and ρ'_S:
  Construct process with $\beta\Delta Q \;\to\; \Delta S \equiv S(\rho_S) - S(\rho'_S)$!
  $\Rightarrow$ need $d \to \infty$

Explicit process next:

- iterated SWAP processes (for given ρ_S, ρ'_S)

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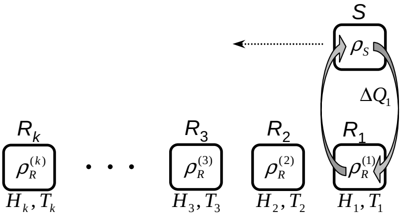
Iterated SWAP process



- $\Delta S_i \;=\; S(\rho^{(i-1)}) - S(\rho^{(i)})$
- $\beta\Delta Q_i \;=\; \Delta S_i + D\left(\rho^{(i-1)} \| \rho^{(i)}\right)$
- $\Delta S \;=\; S(\rho^{(0)}) - S(\rho^{(k)}) \;=\; S(\rho_S) - S(\rho'_S)$

$$\beta\Delta Q \;=\; \Delta S + \sum_{i=1}^{k} D\big(\rho^{(i-1)} \| \rho^{(i)}\big) \gtrsim \Delta S + \frac{(\Delta S)^2}{k \cdot \log^2 d}$$

energy–time tradeoff ? attainable ? thdyn. “reversibility” ?

Iterated SWAP process

$$\sum_{i=1}^{k} D\big(\rho^{(i-1)} \| \rho^{(i)}\big) = -\sum_{i=1}^{k} \mathrm{tr}\big[\rho^{(i-1)}\big(\log\rho^{(i)} - \log\rho^{(i-1)}\big)\big]$$

$$\overset{(k\to\infty)}{\to} -\sum_{i=1}^{k} \mathrm{tr}\Big[\rho^{(i-1)}\Big(\big(\rho^{(i-1)}\big)^{-\frac{1}{2}}\delta\rho_i\big(\rho^{(i-1)}\big)^{-\frac{1}{2}}\Big)\Big] = 0.$$

$\Rightarrow$ Landauer bound sharp with $d = d_S^k \to \infty$

- Anders/Giovannetti (2012): $\rho^{(i)} := \frac{i}{k}\rho'_S + \frac{k-i}{k}\rho_S$. Then:

$$\sum_{i=1}^{k} D\big(\rho^{(i-1)} \| \rho^{(i)}\big) \leq \frac{D(\rho_S\|\rho'_S) + D(\rho'_S\|\rho_S)}{k}$$

$\to$ matches our lower bound $\gtrsim (\Delta S)^2/k$ for k swaps
$\to$ bound for general processes: $\geq M(\Delta S, d_S^k) \gtrsim (\Delta S)^2/k^2$

Caveat: What if $\mathrm{rank}(\rho'_S) < \mathrm{rank}(\rho_S)$?

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Conclusion & technical questions

- minimal assumptions: $\rho_S \otimes e^{-\beta H} \quad \overset{U}{\mapsto} \quad \rho'_{SR}$
- LP equality: $\beta\Delta Q \;=\; \Delta S \;+\; I(S' : R') \;+\; D(\rho'_R \| \rho_R)$
- finite-size effects: $\beta\Delta Q \;\geq\; \Delta S + \frac{(\Delta S)^2}{2\log^2 d}$
  - $10\% - 50\%$ for reservoir of $N \leq 5$ qubits
  - model for energy-time tradeoff

- tight bound for $\Delta S < 0$? [possibly $\Delta S < -\log d$]
- take $I(S' : R')$-term into account
- $\beta\Delta Q \;\geq\; F(\rho_S, \rho'_S, d)$
- formulation & proof for C^*-dynamical reservoir/system

Thermodynamics of information processing

von Neumann (1949): computers, biological systems, brain: $\gtrsim k_B T$ heat for *every* computing operation

Landauer (1961):
– heat dissipation for *logically irreversible* operations, e.g. erasure
– $k_B T \log 2$ per bit erased
– justification by 2^{nd} Law

Bennett (1973):
– reversible computation (esp. *quantum*)
– but prone to error ($\to$ error correction)
– energy expense for error correction
$\to$ year 2000: $\sim 500\, k_B T$ per bit

Maxwell (1871): Maxwell's Demon
Szilard (1929): Szilard engine

Why erasure?

- computation result / error syndrome "0" /"1" in register S
- "unknown" to outside: $\rho_S = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $S(\rho_S) = 1$ bit
- before next computation / error correction:

  RESET: $\rho_S = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \mapsto \quad \rho'_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- cannot do it reversibly:

  $$|0\rangle_S|0\rangle_M \mapsto |0\rangle_S|0\rangle_M \mapsto |0\rangle_S|0\rangle_M$$
  $$|1\rangle_S|0\rangle_M \mapsto |1\rangle_S|1\rangle_M \mapsto |0\rangle_S|0\rangle_M \text{ ???}$$

  another way to see this: unitaries preserve spectrum
- $\rightarrow$ need resource: e.g. reservoir at temperature T
- more generally: "replace" ρ_S by any given ρ'_S

Derivation: using 2^{nd} Law (Landauer, . . .)

1. want: possible states “0” / “1” $\mapsto$ definite final state “0”
2. $\Rightarrow$ S_{system} decreases by $\log 2$
3. $\Rightarrow$ by 2^{nd} Law: $S^{final}_{environ} - S^{initial}_{environ} \geq \log 2$
4. $\Rightarrow$ by Th-dynamics: $\Delta Q_{environ} \geq k_B T \log 2$.

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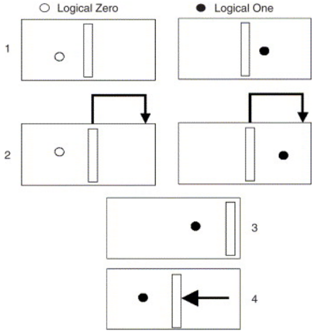
more generally: all logically irreversible operations

- bit erasure: $a \mapsto 0$
- AND: $(a, b) \mapsto (a, a \,\text{AND}\, b)$
- OR: $(a, b) \mapsto (a, a \,\text{OR}\, b)$
- . . .

Derivation: 1-particle gas



3 → 4: isothermal compression

$$
\begin{aligned}
\Delta Q &= -\Delta W \\
&= -\int_V^{V/2} p(V')dV' \\
&= -\int_V^{V/2} \frac{1 \cdot k_B T}{V'} dV' \\
&= k_B T \log 2 \ .
\end{aligned}
$$

(if “quasi-static”!)

S = information-bearing d.o.f.

E = velocity, exact location, ...

Pureness of final state

$$\lambda_{min}(\rho'_S) \;\geq\; \sum_{i=1}^{d} \lambda_i^{\uparrow}(\rho'_{SR}) \;=\; \sum_{i=1}^{d} \lambda_i^{\uparrow}(\rho_S \otimes \rho_R) \;\geq\; d\,\lambda_{min}(\rho_S)\,\lambda_{min}(\rho_R)$$

$$\lambda_{min}(\rho_R) \;=\; \frac{e^{-\beta H_{max}}}{\mathrm{Tr}\left[e^{-\beta H}\right]} \;\geq\; \frac{e^{-\beta H_{max}}}{d\,e^{-\beta H_{min}}}$$

$$\frac{\lambda_{min}(\rho'_S)}{\lambda_{min}(\rho_S)} \;\geq\; e^{-\beta(H_{max}-H_{min})} \;\geq\; e^{-2\beta\|H\|}\,.$$

$\rightarrow$ “To erase 1 qubit”, need:

- zero-temperature reservoir ($\beta = \infty$, i.e. $\beta\Delta Q = \infty$)
- formally: $H_{max} = +\infty$

  in this case: $d < \infty \;\Rightarrow\; \Delta Q = \infty$

  $d = \infty \;\rightarrow\;$ later

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Example 2: Erasure towards pure states

- Want: $\rho_S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \overset{\text{process}}{\mapsto} \rho_S' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- $\rho_S \otimes \rho_R \mapsto \rho_S' \otimes \rho_R'$ preserves rank
  $\Rightarrow \beta\Delta Q = \infty$ whenever $\dim(R) < \infty$
- For $d = \infty$: permutation of product eigenstates
  $\rho_R = (0, r_1, 0, r_2, 0, r_3, 0, r_4, 0, r_5, 0, \ldots)$ over ℓ^2
  $\rho_R' = (0, s_1 r_1, 0, s_2 r_1, 0, s_1 r_2, 0, s_2 r_2, 0, s_1 r_3, \ldots)$

  Choose: $r_1 = 0,\ r_3 = (1-\varepsilon)s_1 r_2,\ r_4 = (1-\varepsilon)s_2 r_2, \ldots$
  $\Rightarrow D(\rho_R' \| \rho_R) = -\log(1-\varepsilon)$

$\Rightarrow$ attain Landauer limit arbitrarily closely
but need $d = \infty$ and infinitely many $+\infty$ energy levels.