Complex quantum states

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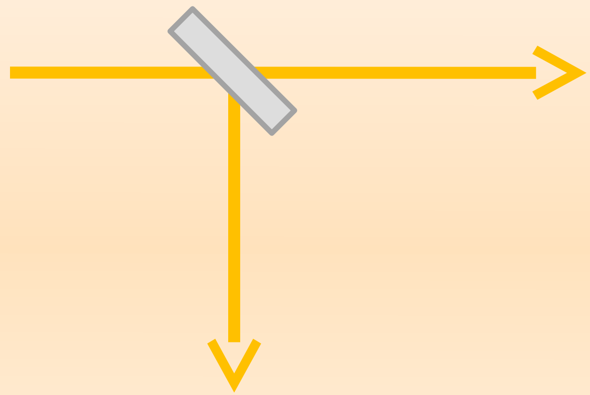
motivation

Schrödinger cat?

“Superposition of two macroscopically distinguishable states”

Example:
A single photon on a beam-splitter



- “To be transmitted” and “to be reflected” are classical alternatives
- There are detectors that can distinguish those alternatives
- By arranging an interferometer, we know that the beam-splitter does create superposition

But this is not exactly what you had in mind, right?

Bigger kittens

Quantum optics: $|+\alpha\rangle + |-\alpha\rangle$ with α "λαργε"

Monomode; not really achieved yet

Heavy weights: $\psi_1 + \psi_2$ for massive objects

Big molecules, nanomechanical oscillators, current in SQUIDS…[*]
Also single degree of freedom

Qubit lover $|0\rangle^{\otimes N} + |1\rangle^{\otimes N}$

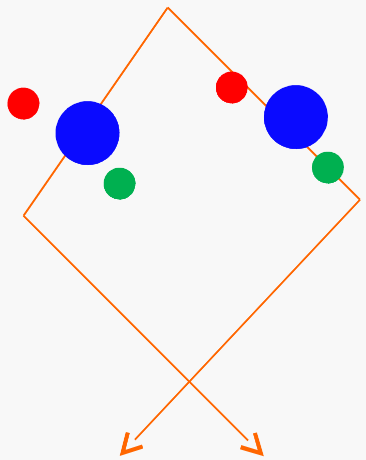
Many degrees of freedom, lots of entanglement… but very easy to describe

[*] "we model the cat as a homogeneous sphere of water with a mass of 4kg"
Nimmrichter and Hornberger, PRL 2013, last sentence of Supplementary Information

A frequent confusion

"Interference of large molecules = GHZ state", because

$$\psi_R + \psi_L = |\vec{x}_R\rangle|\vec{x}_R + \vec{r}_1\rangle|\vec{x}_R + \vec{r}_2\rangle \ldots + |\vec{x}_L\rangle|\vec{x}_L + \vec{r}_1\rangle|\vec{x}_L + \vec{r}_2\rangle \ldots$$



Disproof:

1) Write in relative coordinates (valid degrees of freedom)

$$\psi_R + \psi_L = (|\vec{x}_R\rangle + |\vec{x}_L\rangle)\ \ |\vec{r}_1\rangle|\vec{r}_2\rangle \ldots$$

2) Notice that full purity is not needed for CM interference:

$$\psi_R + \psi_L = (|\vec{x}_R\rangle + |\vec{x}_L\rangle) \otimes \rho$$

(Sorry for the sloppy notation)

Complexity 101

?

- States useful for quantum computing must be "complex"
  - Simple states are easy to simulate
  - Critics of QC base their skepticism on the very possibility of creating such states
- Biological systems are also complex (OK, and in a hot environment too)

The Schrödinger cat is probably complex

- "Complexity is not computable"
  - Kolmogorov complexity is indeed not; other measures are: see next
- "Complex states are not feasible with current technologies"
  - True. But aren't you tired of stuff that is "feasible with current technologies"? Shouldn't theorists look a bit further ahead?

first steps in the forest

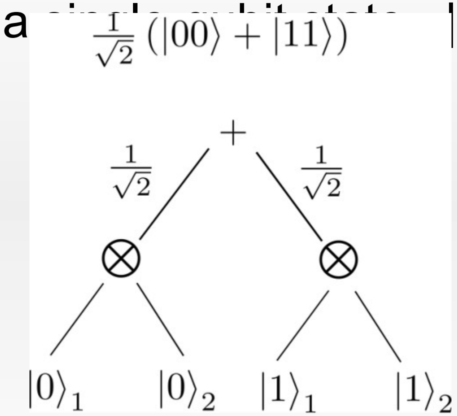
The “tree size” of a quantum state

Tree size of a quantum state

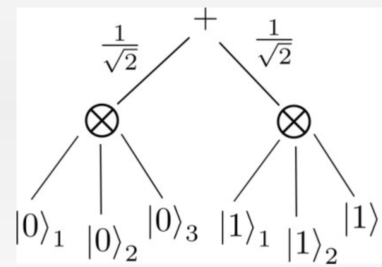
[Proposed by Aaronson STOC'04]

Any multiqubit quantum state can be described by a rooted tree of $\otimes$ and + gates

Each leaf is labeled with a single qubit state $\alpha|0\rangle+\beta|1\rangle$



$|GHZ\rangle$



- Size of a tree = **number of leaves.**
- **Tree size of a state (TS) = size of the *minimal* tree = most compact way of writing the state**

$$\underbrace{|0\rangle|0\rangle+|0\rangle|1\rangle+|1\rangle|0\rangle+|1\rangle|1\rangle}_{\text{8 leaves}} = \underbrace{|+\rangle|+\rangle}_{\text{2 leaves}}$$

Getting compact (example: 3 qubits)

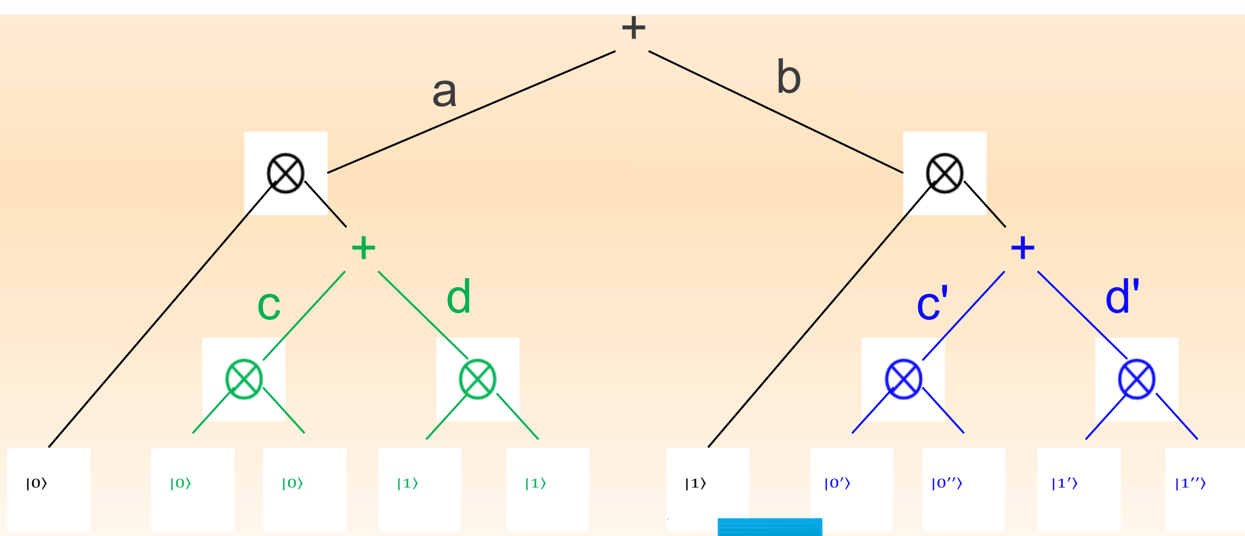
Naïve computational ba

$$\sum_{x_i=0,1} c_{x_1x_2x_3}|x_1\rangle|x_2\rangle|x_3\rangle$$

24 leaves

Sequential Schmidt:

$$a|0\rangle(c|0\rangle|0\rangle + d|1\rangle|1\rangle) + b|0\rangle(c'|0'\rangle|0''\rangle + d'|1'\rangle|1''\rangle)$$



10 leaves

Using Acin et al. 2001:

$$a|0\rangle|0\rangle|0\rangle + b|1\rangle(c|0'\rangle|0''\rangle + d|1'\rangle|1''\rangle)$$

8 leaves

Examples and bounds

$TS(|0\rangle^{N}) = n$

$TS(GHZ) = 2n$ (I told you it's simple)

$TS(W) = O(n^2)$

$TS(1D\ cluster) = \cancel{O(n^4)}$

$O(n^2)$ this Monday's result ☺

$TS(2D\ cluster) = 2^{\Omega(n)}$ conjectured

$TS(Shor) = n^{\Omega(\log n)}$ proved under one conjecture

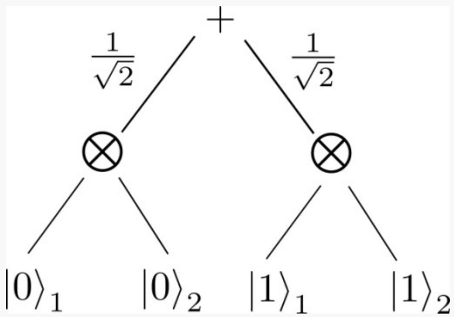
Anything provable somewhere here?

- Upper bound on TS: nested Schmidt decomposition (or slightly more clever one) ⇒ easy to prove that some states are NOT complex
- Conjectures: basically, if it allows universal QC, it cannot be too simple, otherwise we could simulate it.
- **Lower** bound on TS: size of a multilinear formula: see next

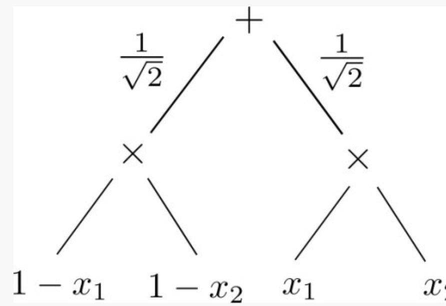
Multilinear formula

$$\Psi = \sum_{x_j=0,1} f(x_1, \ldots, x_n)|x_1, \ldots, x_n\rangle$$

We want a multilinear formula to compute the coefficients



$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



$\frac{1}{\sqrt{2}}[(1 - x_1)(1 - x_2) + x_1 x_2]$

So, what have we gained?

We can stand on the shoulder of mathematicians!

Raz, STOC'04: any multilinear formula that computes the **determinant** or **permanent** of a matrix is **super-polynomial**. So…

States with super-polynomial TS

arXiv:1303.4843

Take n=m2 qubits, and arrange the coefficients as a matrix

$$\{x\} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{pmatrix}.$$

Then these states have super-polynomial TS:

$$|\mathrm{det}_m\rangle = \sum_{x=0}^{2^n-1} \det(\{x\})\,|x\rangle\,,$$

$$|\mathrm{per}_m\rangle = \sum_{x=0}^{2^n-1} \mathrm{perm}(\{x\})\,|x\rangle$$

Raz

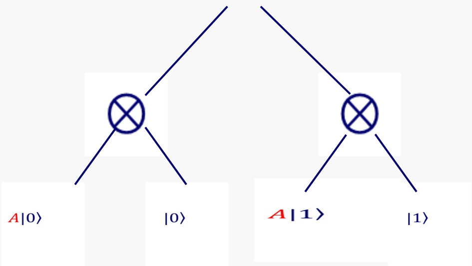
Our best decomposition

$$n^{\Omega(\log n)/4} \leq TS \leq O((\sqrt{n})!)$$

Similar to the bound "proved" for Shor

Work in progress (1): tight bounds

Tools: one can use the SLOCC classification, since the tree size does not change for a reversible LOCC:



3 qubits

- Biseparable: TS = 5
- GHZ class: TS = 6
- W class: TS = 8

Most complex, but of zero measure:
$|W\rangle \to |W\rangle + \epsilon|111\rangle \in GHZ$

4 qubits

- Several classes
- $TS \leq 14$ because any state can be written as $|0\rangle|GHZ\rangle + |1\rangle|GHZ'\rangle$ up to SLOCC
- There are states with TS=14 and their set is not of zero measure. E.g. $|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle$

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Work in progress (2): TS & MPS

Assume for simplicity **n=2k** qubits:

$$|MPS_d^n\rangle = \underbrace{(A_0{}^1|0\rangle + A_1{}^1|1\rangle)}_{1\mathrm{x}d} \prod_{j=2}^{n-1} \underbrace{(A_0{}^j|0\rangle + A_1{}^j|1\rangle)}_{d\mathrm{x}d} \underbrace{(A_0{}^n|0\rangle + A_1{}^n|1\rangle)}_{d\mathrm{x}1}$$

$$= \sum_{i=1}^{D} |MPS_{d,i}^{n/2}\rangle |MPS_{d,i}^{n/2}\rangle$$

Proof: insert $I_{d\times d} = e_1 e_1^T + \cdots + e_d e_d^T$

So $\mathrm{TS_k} \leq 2d \times TS_{k-1} \leq \cdots \leq (2d)^k$

that is $\mathrm{TS_{n,d}} \leq (2d)^{\log_2 n} = n^{\log_2 2d}$

- If d constant, TS polynomial in n
- 1D cluster state: d=2, so TS=O(n2).

Missing links

Wanted: operational interpretation(s)

- Links with measures of "difficulty"
  - Some polynomial circuit can generate superpolynomial TS states with finite probability
  - Rk: Shor's algorithm also has polynomial circuit size…
  - No link known with Hamiltonian families (beyond indirect ones through MPS, see above)
- Link with universal quantum computing?
  - Clearly small TS $\Rightarrow$ easy to describe $\Rightarrow$ not useful
- Does high TS mean "harder to produce in the lab" even for few qubits?
- "Natural" situations in which such states appear
  - Biological systems??

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Trying to enthuse Jose-Ignacio

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