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Optimal discrimination of quantum measurements

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joint work with
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Outline

- State discrimination approaches
- Definition of the investigated problem
- Possible solutions and their optimality
- Mathematical framework of PPOVMs
- Discrimination of qubit Von Neuman measurements
- Results and consequences

Acknowledgments:

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european social fund in the czech republic

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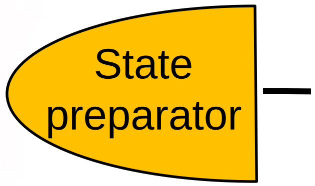
OP Education for Competitiveness

INVESTMENTS IN EDUCATION DEVELOPMENT

State discrimination

- A single copy of a state is know to be either ψ or ξ, with prior probability η_ψ , η_ξ



- Goal: distinguish the two possibilities

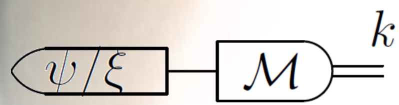
Known fact:

Non orthogonal quantum states cannot be perfectly discriminated

Simplest problem:

Quantum states ψ , ξ are pure

Minimum error discrimination

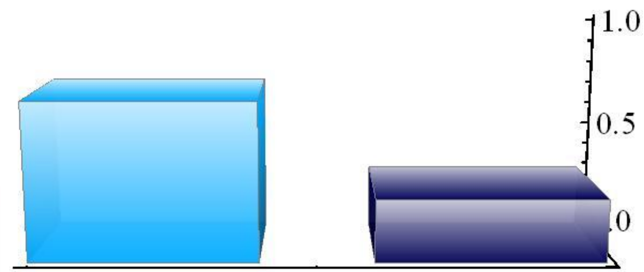


Outcome of the measurement

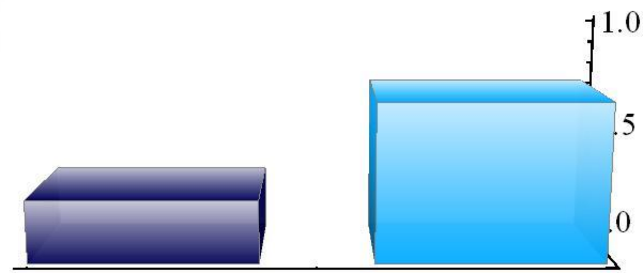
1 ≡ guess ψ 2 ≡ guess ξ

Tested state

ψ prepared



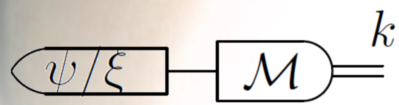
ξ prepared



$$p_e = \eta_\psi \, p(2|\psi, \mathcal{M}) + \eta_\xi \, p(1|\xi, \mathcal{M})$$

$$p_s = 1 - p_e$$

Unambiguous discrimination



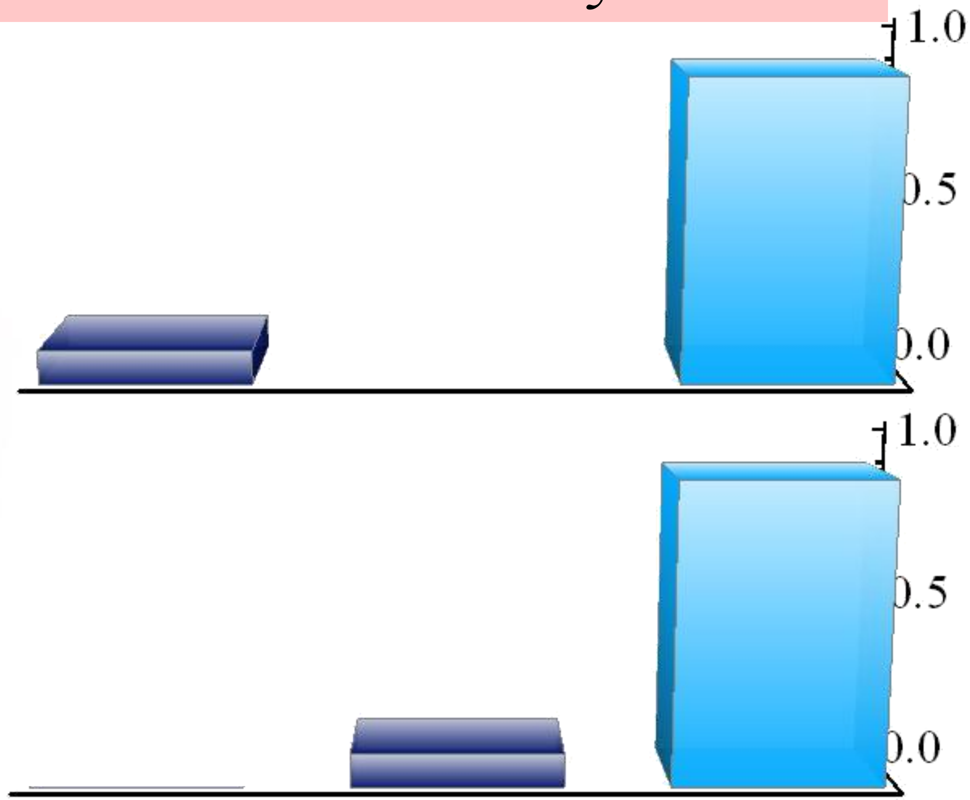
Outcome of the measurement

$1 \equiv$ guess ψ $2 \equiv$ guess ξ $0 \equiv$ failure

Tested state

ψ prepared

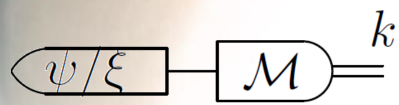
ξ prepared



$$p_e = 0$$
$$p_s = 1 - p_f$$

$$p_f = \eta_\psi \, p(0|\psi, \mathcal{M}) + \eta_\xi \, p(0|\xi, \mathcal{M})$$

Discrimination with error margin *m*



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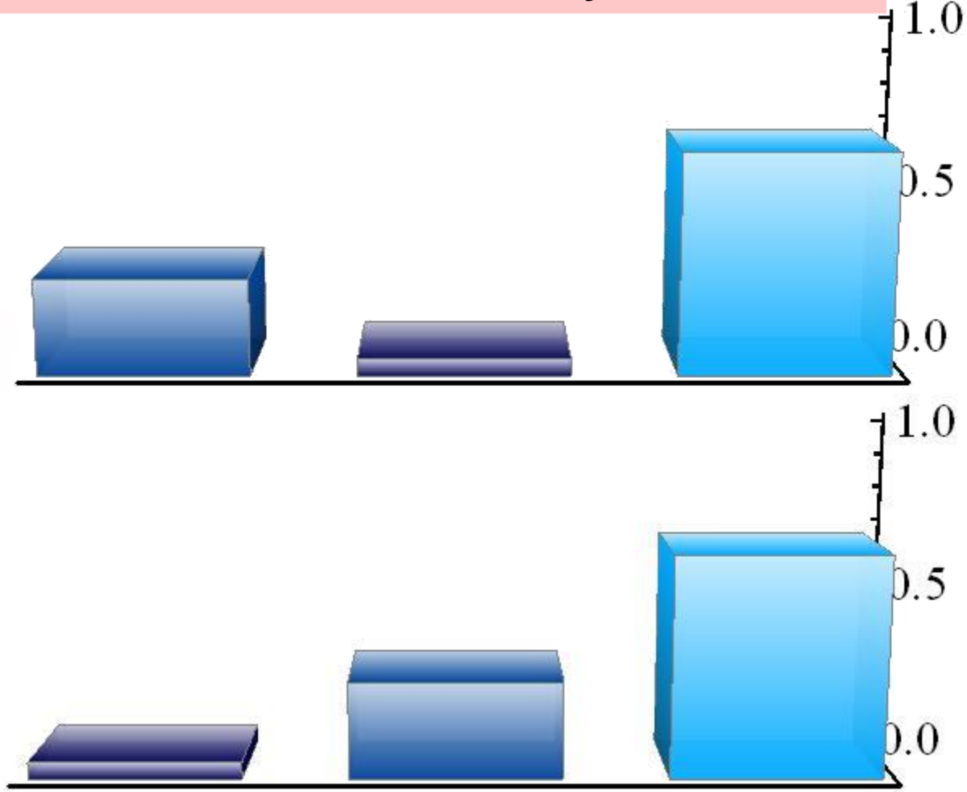
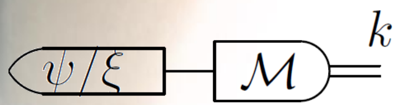


- **Highest probability of success for a fixed probability of error**

Discrimination with error margin *m*



Outcome of the measurement

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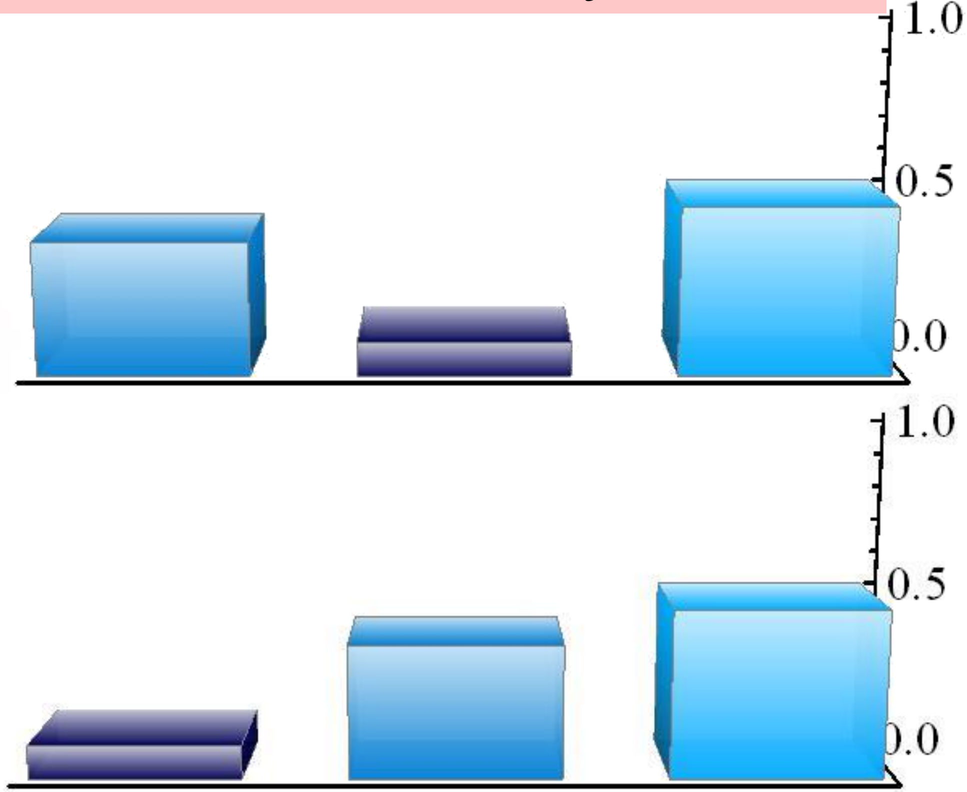
Tested state

ψ prepared

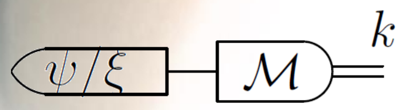
ξ prepared



We require:
$p_e \leq m$

We maximize: $p_s = \eta_\psi \, p(1|\psi, \mathcal{M}) + \eta_\xi \, p(2|\xi, \mathcal{M})$

Discrimination with error margin *m*



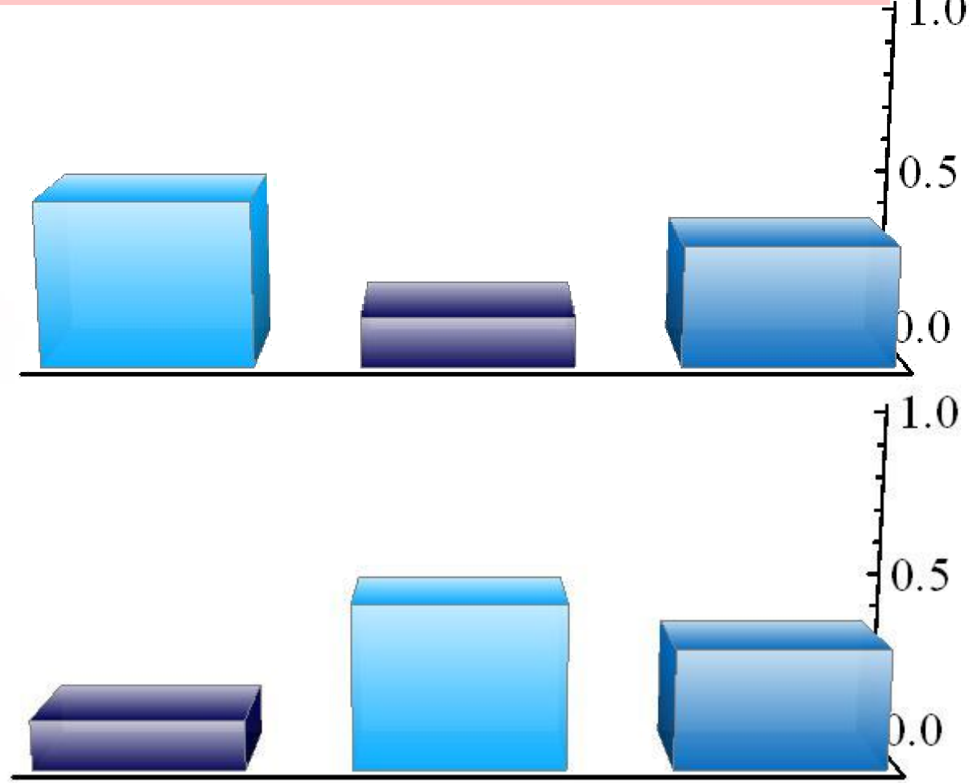
Outcome of the measurement

1 ≡ guess ψ 2 ≡ guess ξ 0 ≡ failure

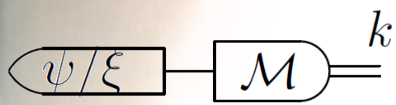
Tested state

ψ prepared

ξ prepared



Discrimination with error margin *m*



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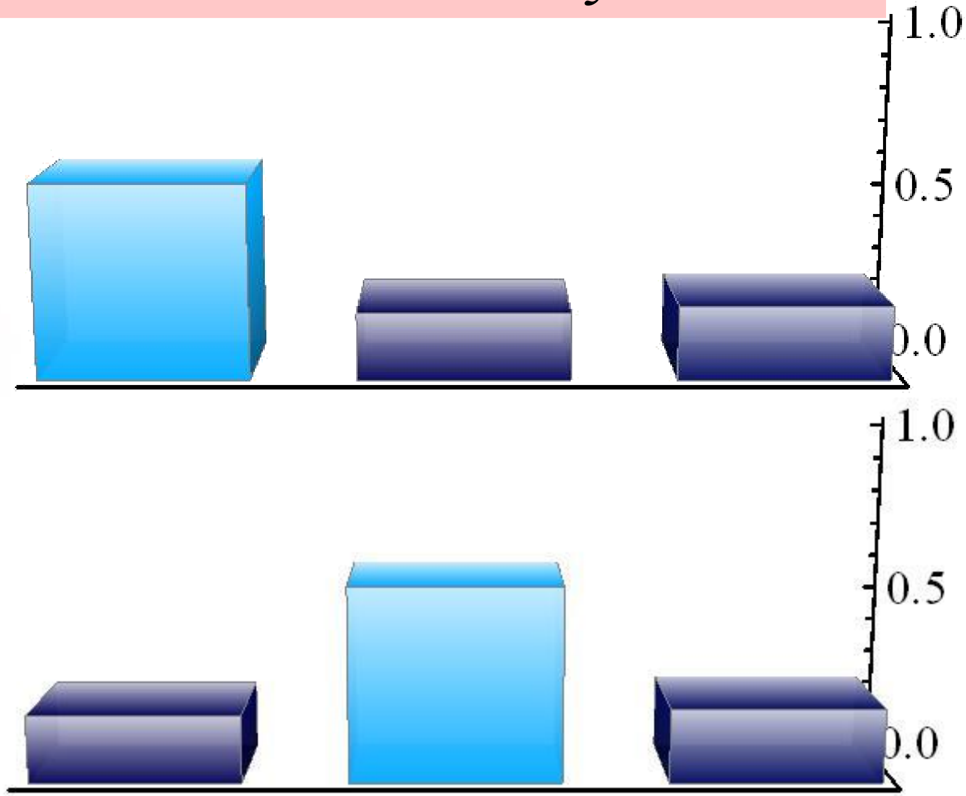
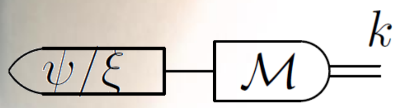


Discrimination with error margin *m*

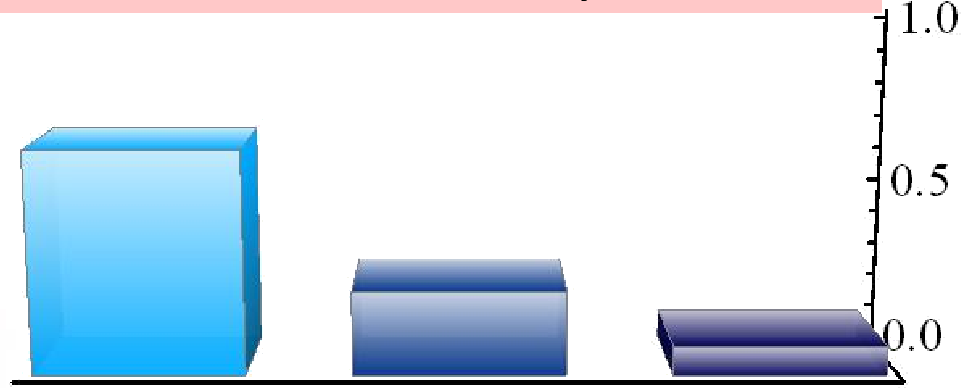


Outcome of the measurement

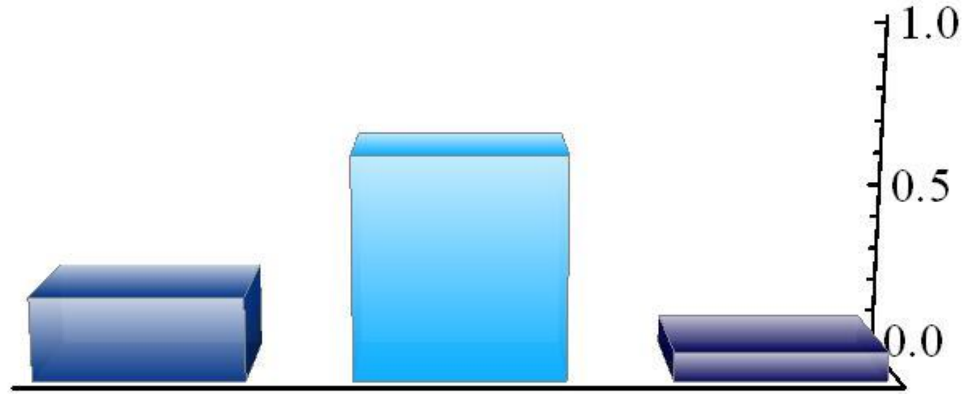
1 ≡ guess ψ 2 ≡ guess ξ 0 ≡ failure

Tested state

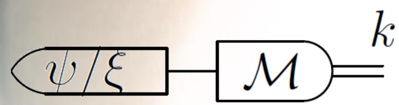
ψ prepared



ξ prepared



Discrimination with error margin *m*



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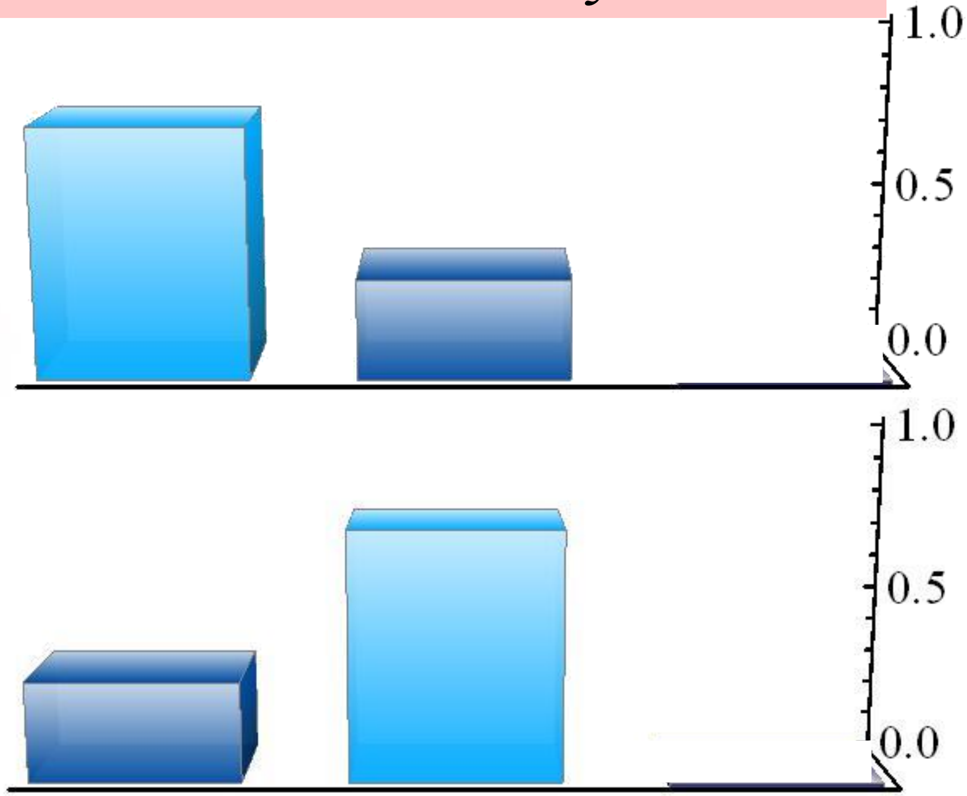


$m = 0$ ⟷ Unambiguous

$m \geq \frac{1}{2}\left(1 - \sqrt{1 - 4\eta_\psi \eta_\xi |\langle\psi|\xi\rangle|^2}\right)$ ⟷ Minimum error

State discrimination - Generalizations

- Discrimination of N pure states

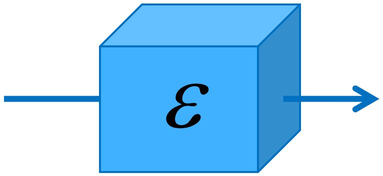
Approach	Status
Minimum error	Known N&S conditions
Unambiguous	Recently solved for N=3
Discrimination with error margin	Unknown

- Discrimination of two mixed states

Approach	Status
Minimum error	Solved
Unambiguous	Still partially open
Discrimination with error margin	only a bound exists

- Numerical aproaches: All these problems can “effectively” treated using Semidefinite programming.

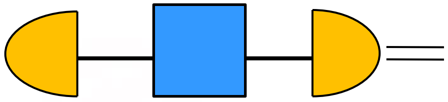
Quantum channel discrimination



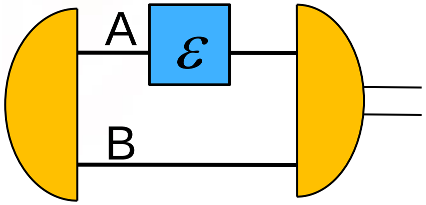
Quantum channel:

- The most general time evolution of a system
- Linear, **completely positive** and trace preserving mapping on $L(\mathcal{H}_A)$

Possible testing schemes:



Simple scheme



Ancilla assisted scheme

Channel discrimination – know facts

- Discrimination of two unitary channels

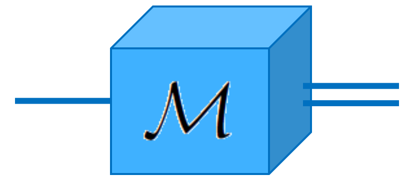
Approach	Status
Minimum error	Known
Unambiguous	Known
Discrimination with error margin	Known

- Discrimination of two arbitrary channels

Approach	Status
Minimum error	Unsolved
Unambiguous	Unsolved
Discrimination with error margin	Unsolved

- For special types of channels minimum error discrimination is solved, conditions for unambiguous discriminability exist

Discrimination of measurements



Quantum measurement:

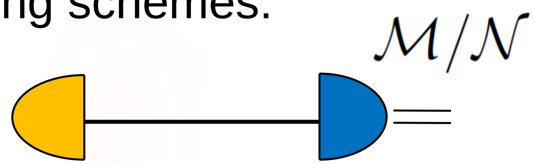
- A device that accepts a quantum system and produces only a classical signal
- Mathematically we describe it by Positive Operator Value Measure (POVM)

$$\mathcal{M} \leftrightarrow \{M_i\}_{i=1}^{r}$$

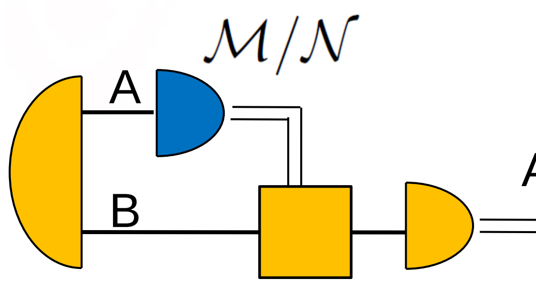
$$\sum_i M_i = I$$

$$M_i \geq 0$$

Possible testing schemes:



Simple scheme



Ancilla assisted schemes

Discrimination of measurements

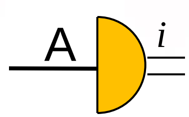
Challenge:

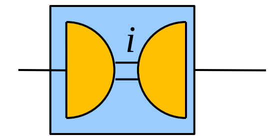
- efficient characterization of most general testing scheme
- Simultaneous optimization of the parameters

Solution:

Mathematically treat measurement as a measure&prepare channel and use the tools for channels.

POVM $\{M_i\}_{i=1}^{r}$ on H_A $\leftrightarrow$ **channel** from H_A into $\mathbb{C}_r$, where outcome i is encoded into ON states





$$\mathcal{M}(\rho) = \sum_i |i\rangle\langle i| Tr(\rho M_i)$$

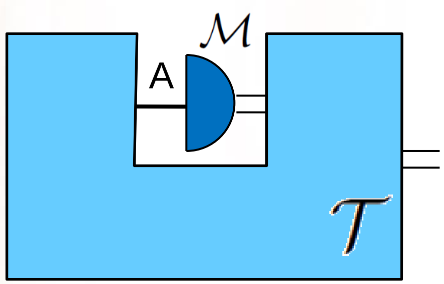
Mathematical framework of Process POVM

- The tested channel is described in CHOI-JAMIOLKOWSKI isomorphism:

$$M = \mathcal{M} \otimes I(|\Omega\rangle\langle\Omega|) = \sum_i |i\rangle\langle i| \otimes M_i^T \qquad |\Omega\rangle = \sum_i |i\rangle|i\rangle$$

- A test of a channel is described by Process POVM:

$$\mathcal{T} = \{T_c\} \qquad T_c \in \mathcal{L}(\mathbb{C}_r \otimes \mathcal{H}_A)$$



$$T_c \geq 0$$

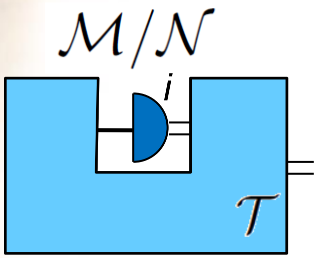
$$\sum_c T_c = I \otimes \rho_A,$$

$$p(c|\mathcal{M}, \mathcal{T}) = Tr(T_c^T M)$$

Process POVM $\mathcal{T}$ describes probabilities of outcomes c obtained in a test of a channel $\mathcal{M}$

Optimal discrimination of measurements

- Equally as for state or channel discrimination we define:

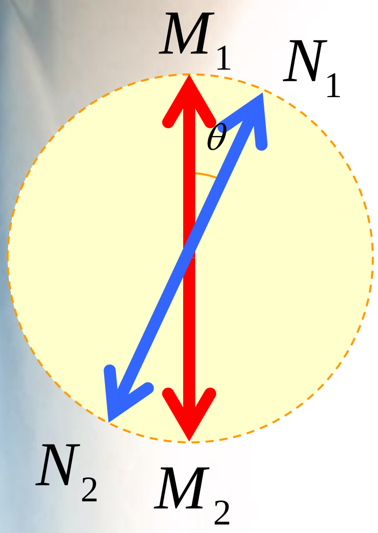


$$p_c = \eta_{\mathcal{M}}\, p(m|\mathcal{M},\mathcal{T}) + \eta_{\mathcal{N}}\, p(n|\mathcal{N},\mathcal{T})$$
$$p_e = \eta_{\mathcal{M}}\, p(n|\mathcal{M},\mathcal{T}) + \eta_{\mathcal{N}}\, p(m|\mathcal{N},\mathcal{T})$$
$$p_f = \eta_{\mathcal{M}}\, p(f|\mathcal{M},\mathcal{T}) + \eta_{\mathcal{N}}\, p(f|\mathcal{N},\mathcal{T}).$$

- The conclusion one can make in general:

Discrimination of measurements can be seen as multiple discrimination of states that are interlinked by the normalization of the Process POVM

Qubit Von Neumann measurements



Two projective measurements to be discriminated:

$$M_1 = |\varphi\rangle\langle\varphi| \qquad M_2 = I - M_1 = |\varphi^\perp\rangle\langle\varphi^\perp|$$

$$N_1 = |\psi\rangle\langle\psi| \qquad N_2 = I - N_1 = |\psi^\perp\rangle\langle\psi^\perp|$$

Thanks to block diagonal structure of M, N
it suffice to consider

$$M = \sum_i |i\rangle\langle i| \otimes M_i^T$$

$$T_c = \sum_i |i\rangle\langle i| \otimes H_i^{(c)}$$

$$c \in \{m, n, f\}$$

Problem can be reformulated as:

$$maximize \quad p_s = \sum_i Tr(\eta_{\mathcal{M}} H_i^{(m)} M_i + \eta_{\mathcal{N}} H_i^{(n)} N_i),$$

$$for \quad p_e \leq m$$

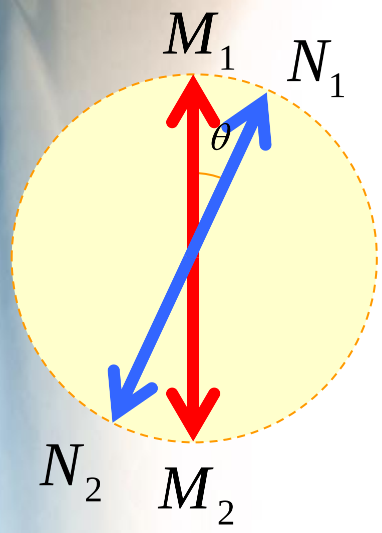
$$\forall i \quad H_i^{(m)} + H_i^{(n)} + H_i^{(f)} = \rho$$

$$Tr(\rho) = 1$$

$$H_i^{(c)} \geq 0$$

$$i = 1, 2$$

Qubit Von Neumann measurements



$M_1 = |\varphi\rangle\langle\varphi|$
$N_1 = |\psi\rangle\langle\psi|$

Universal NOT operation Γ:

$$\Gamma(M_1) = M_2 \quad \Gamma(N_1) = N_2$$

Symmetry

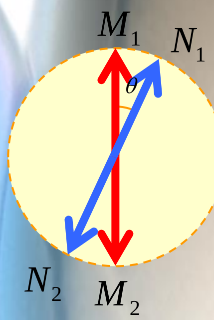
$$H_2^{(c)} = \Gamma(H_1^{(c)}) \qquad \rho = \frac{1}{2} I$$

can be imposed without loss of generality.

The problem is now specified by:

$$\begin{aligned} &maximize \quad p_s = \eta_{\mathcal{M}} \langle\varphi|2H_1^{(m)}|\varphi\rangle + \eta_{\mathcal{N}} \langle\psi|2H_1^{(n)}|\psi\rangle \\ &for \quad p_e \leq m \\ &2H_1^{(m)} + 2H_1^{(n)} + 2H_1^{(f)} = I \qquad H_1^{(c)} \geq 0 \end{aligned}$$

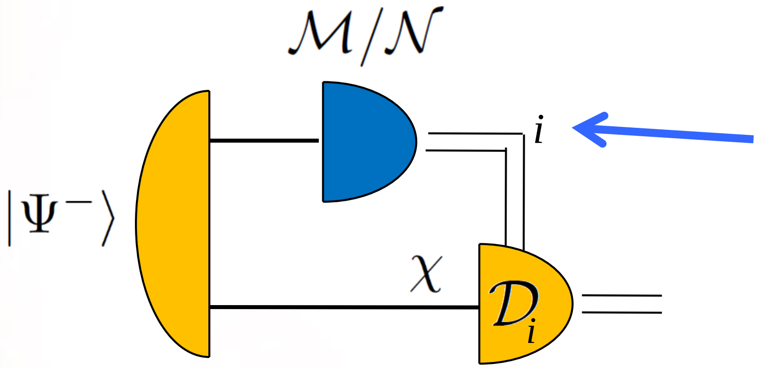
This is formally equivalent to a discrimination of pure states $|\varphi\rangle$ and $|\psi\rangle$ by a POVM $E_c \equiv 2H_1^{(c)}$.



Qubit Von Neumann measurements

Optimal discrimination procedure:

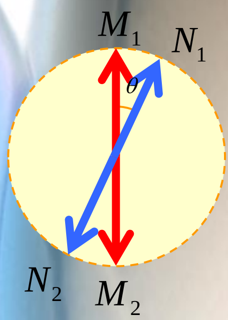
$$M_1 = |\varphi\rangle\langle\varphi|$$
$$N_1 = |\psi\rangle\langle\psi|$$



result i chooses a pair of states which should be optimally distinguished by measurement $\mathcal{D}_i$.

for i =1 the state χ is either $|\varphi^\perp\rangle$ or $|\psi^\perp\rangle$
for i =2 the state χ is either $|\varphi\rangle$ or $|\psi\rangle$

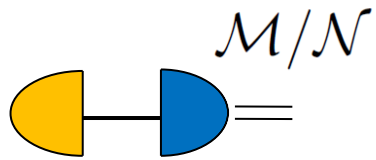
This procedure is optimal for minimum error, unambiguous discrimination and discrimination of measurements with fixed error margin



Qubit Von Neumann measurements

When is a simple discrimination scheme optimal?

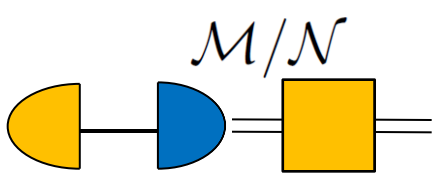
$$M_1 = |\varphi\rangle\langle\varphi|$$
$$N_1 = |\psi\rangle\langle\psi|$$



Simple scheme

Single measurement detection regime
(very unbalanced prior probabilities)
in unambiguous discrimination

Minimum error discrimination of qubit Von Neuman measurements (very unbalanced prior probabilities require classical processing of outcomes)



Our derivation generalizes easily to discrimination of

N - Von Neumann measurements on gubit

- optimization is equivalent to discrimination of the corresponding N pure states defining the measurements

Two noisy qubit measurements

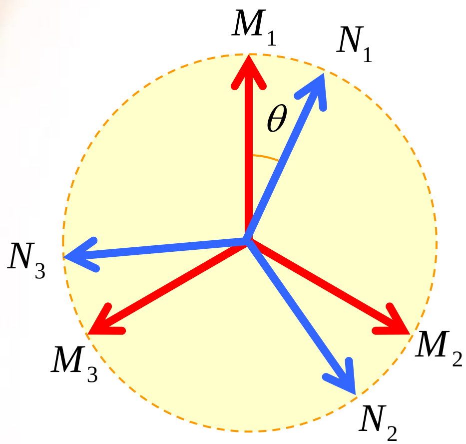
- Von Neumann measurements mixed with white noise

$$M_1 = \delta|\varphi\rangle\langle\varphi| + \frac{1-\delta}{2}I \qquad M_2 = \delta|\varphi^\perp\rangle\langle\varphi^\perp| + \frac{1-\delta}{2}I$$

$$N_1 = \delta|\psi\rangle\langle\psi| + \frac{1-\delta}{2}I \qquad N_2 = \delta|\psi^\perp\rangle\langle\psi^\perp| + \frac{1-\delta}{2}I.$$

- Optimization is equivalent to a discrimination of states defined by operators M_1, N_1 .

Two Trine measurements on a qubit



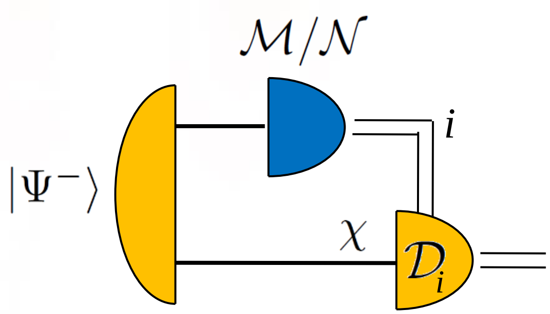
$$M_1 = \frac{2}{3}|0\rangle\langle 0|_Z$$

$$M_2 = R_X[\frac{2\pi}{3}](M_1)$$

$$M_3 = R_X[\frac{4\pi}{3}](M_1)$$

$$N_i = R_X[\theta](M_i) \ \forall i$$

Optimal unambiguous discrimination:

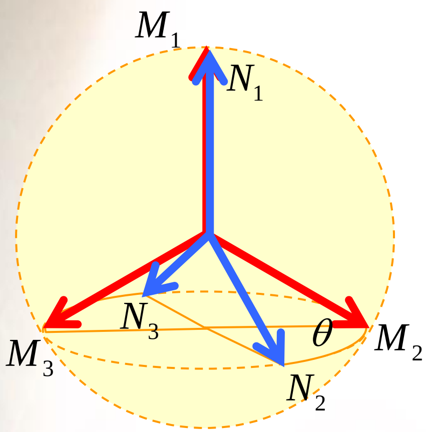


Ancilla assisted scheme with maximally entangled state is optimal

Measurement $\mathcal{D}_i$ discriminates pairs of states differing angle θ

Two Trine measurements on a qubit

Optimal unambiguous discrimination:

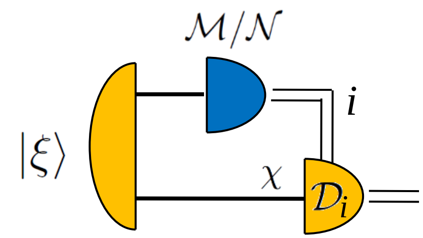


$$M_1 = \frac{2}{3}|0\rangle\langle 0|_Z$$

$$M_2 = R_X[\frac{2\pi}{3}](M_1)$$

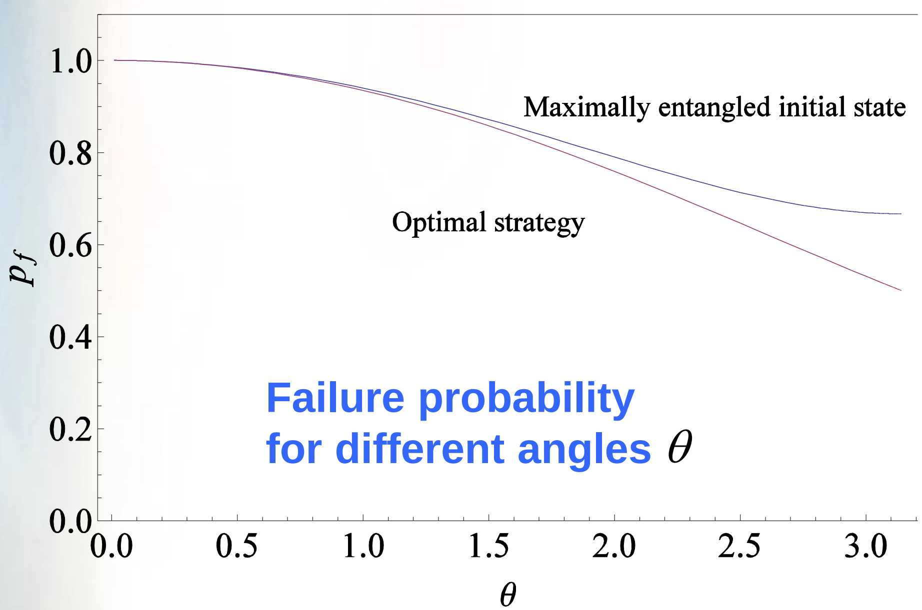
$$M_3 = R_X[\frac{4\pi}{3}](M_1)$$

$$N_i = R_Z[\theta](M_i) \ \forall i$$



$$|\xi\rangle = \lambda|0\rangle|0\rangle + \sqrt{1-\lambda^2}|1\rangle|1\rangle$$

$\lambda = \frac{\sqrt{3}}{2}$ for $\theta = \pi$



Ancilla assisted scheme with **non-maximally** entangled state is optimal

Poster on experiment:

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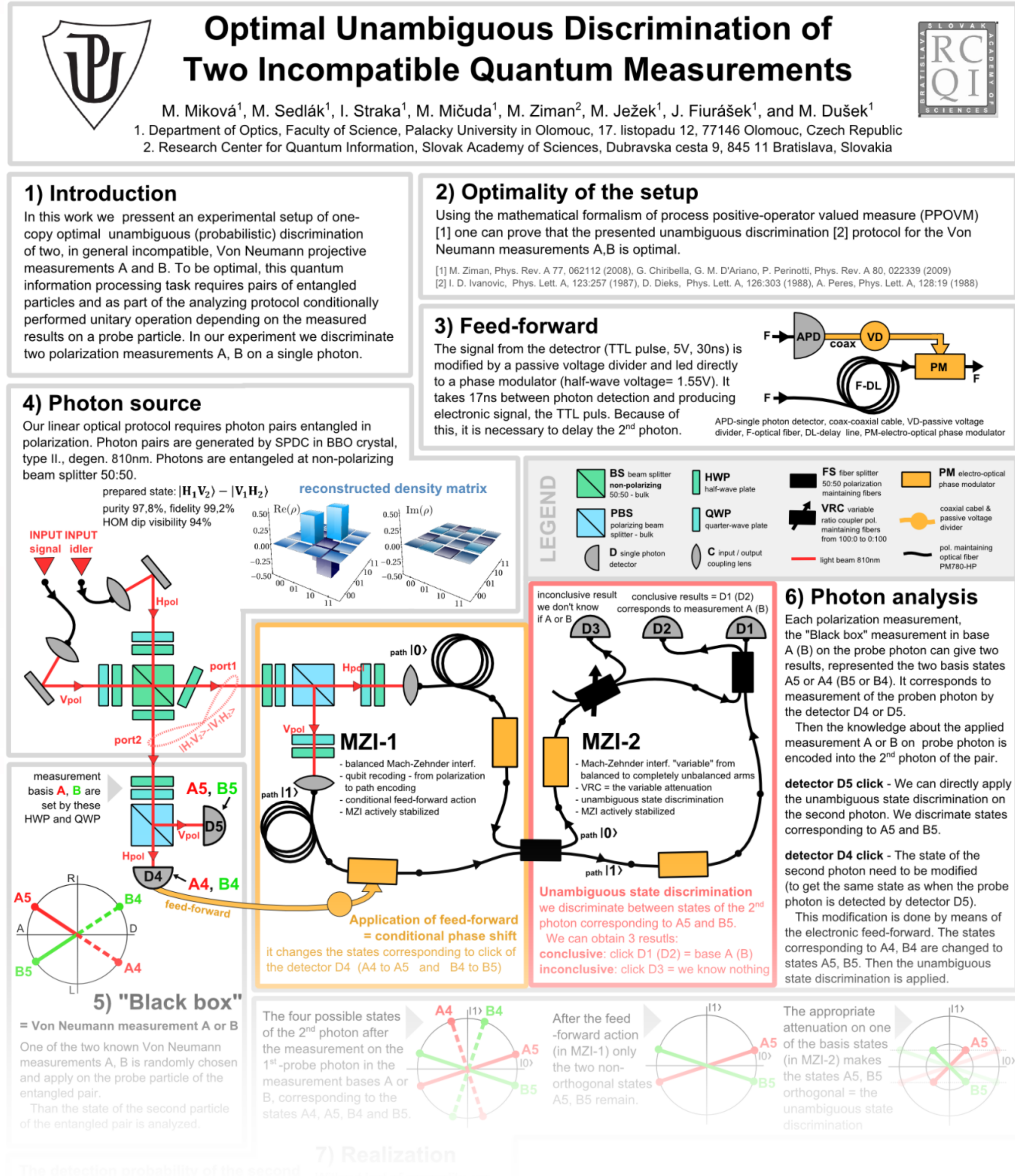

Optimal Unambiguous Discrimination of Two Incompatible Quantum Measurements

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1) Introduction

In this work we pressent an experimental setup of one-copy optimal unambiguous (probabilistic) discrimination of two, in general incompatible, Von Neumann projective measurements A and B. To be optimal, this quantum information processing task requires pairs of entangled particles and as part of the analyzing protocol conditionally performed unitary operation depending on the measured results on a probe particle. In our experiment we discriminate two polarization measurements A, B on a single photon.

2) Optimality of the setup

Using the mathematical formalism of process positive-operator valued measure (PPOVM) [1] one can prove that the presented unambiguous discrimination [2] protocol for the Von Neumann measurements A,B is optimal.

[1] M. Ziman, Phys. Rev. A 77, 062112 (2008), G. Chiribella, G. M. D'Ariano, P. Perinotti, Phys. Rev. A 80, 022339 (2009)
[2] I. D. Ivanovic, Phys. Lett. A, 123:257 (1987), D. Dieks, Phys. Lett. A, 126:303 (1988), A. Peres, Phys. Lett. A, 128:19 (1988)

3) Feed-forward

The signal from the detector (TTL pulse, 5V, 30ns) is modified by a passive voltage divider and led directly to a phase modulator (half-wave voltage= 1.55V). It takes 17ns between photon detection and producing electronic signal, the TTL puls. Because of this, it is necessary to delay the 2nd photon.

APD-single photon detector, coax-coaxial cable, VD-passive voltage divider, F-optical fiber, DL-delay line, PM-electro-optical phase modulator

4) Photon source

Our linear optical protocol requires photon pairs entangled in polarization. Photon pairs are generated by SPDC in BBO crystal, type II., degen. 810nm. Photons are entangeled at non-polarizing beam splitter 50:50.

prepared state: $|H_1V_2\rangle - |V_1H_2\rangle$
purity 97,8%, fidelity 99,2%
HOM dip visibility 94%

reconstructed density matrix

6) Photon analysis

Each polarization measurement, the "Black box" measurement in base A (B) on the probe photon can give two results, represented the two basis states A5 or A4 (B5 or B4). It corresponds to measurement of the proben photon by the detector D4 or D5.

Then the knowledge about the applied measurement A or B on probe photon is encoded into the 2nd photon of the pair.

detector D5 click - We can directly apply the unambiguous state discrimination on the second photon. We discrimate states corresponding to A5 and B5.

detector D4 click - The state of the second photon need to be modified (to get the same state as when the probe photon is detected by detector D5).

This modification is done by means of the electronic feed-forward. The states corresponding to A4, B4 are changed to states A5, B5. Then the unambiguous state discrimination is applied.

MZI-1
- balanced Mach-Zehnder interf.
- qubit recoding - from polarization to path encoding
- conditional feed-forward action
- MZI actively stabilized

MZI-2
- Mach-Zehnder interf. "variable" from balanced to completely unbalanced arms
- VRC = the variable attenuation
- unambiguous state discrimination
- MZI actively stabilized

Application of feed-forward = conditional phase shift
it changes the states corresponding to click of the detector D4 (A4 to A5 and B4 to B5)

Unambiguous state discrimination
we discriminate between states of the 2nd photon corresponding to A5 and B5.
We can obtain 3 resutls:
conclusive: click D1 (D2) = base A (B)
inconclusive: click D3 = we know nothing

5) "Black box"

= Von Neumann measurement A or B

One of the two known Von Neumann measurements A, B is randomly chosen and apply on the probe particle of the entangled pair.

Than the state of the second particle of the entangled pair is analyzed.

The four possible states of the 2nd photon after the measurement on the 1st-probe photon in the measurement bases A or B, corresponding to the states A4, A5, B4 and B5.

After the feed-forward action (in MZI-1) only the two non-orthogonal states A5, B5 remain.

The appropriate attenuation on one of the basis states (in MZI-2) makes the states A5, B5 orthogonal = the unambiguous state discrimination

Summary

- Discrimination of measurements can be seen as multiple discrimination of states that are interlinked by the normalization of the Process POVM

Qubit Von Neumann measurements

- Discrimination is formally equivalent to a discrimination of pure states $|\varphi\rangle$ and $|\psi\rangle$ by a POVM.
- In general ancila assisted test with feed-forward of the measurement outcome is needed
- The optimal probe state is not always maximally entangled
- For minimum error discrimination just simple test procedure suffices

Open questions:

- How to solve discrimination between two Von Neuman measurements on a qudit

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Thanks for your attention.