Zero-error communication: the role of non-local resources

Andreas Winter
(ICREA & Universitat Autònoma de Barcelona)
[with T.S. Cubitt, R. Duan, D. Leung, W. Matthews and S. Severini]

Outline

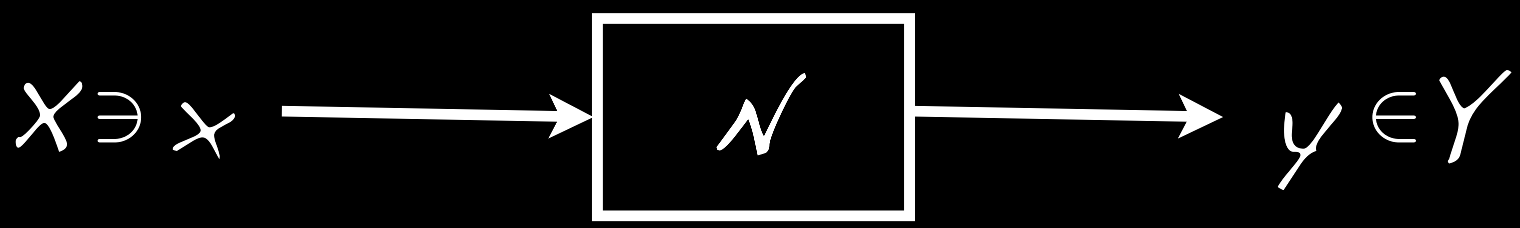
1. Channels and their graphs
2. Zero-error comm. with non-local resources
3. Operator characterization of $\tilde{\alpha}$, Beigi's bound and the Lovász number
4. Entanglement can make a difference
5. Quantum channel generalizations

Outline

1. Channels and their graphs
2. Zero-error comm. with non-local resources
3. Operator characterization of $\tilde{\alpha}$, Beigi's bound and the Lovász number
4. Entanglement can make a difference
5. Quantum channel generalizations NEW

1. Channels & graphs

Channel $N : X \longrightarrow Y$ = stochastic map



$N(y|x)$: transition probabilities

(i.e., $Y \times X$-matrix).

1) Transition graph Γ: bipartite graph on $X \times Y$ with adjacency matrix

$$\Gamma(y|x) = \begin{cases} 1 & \text{if } N(y|x) > 0, \\ 0 & \text{if } N(y|x) = 0. \end{cases}$$

1) Transition graph Γ: bipartite graph on $X \times Y$ with adjacency matrix

$$\Gamma(y|x) = \begin{cases} 1 & \text{if } N(y|x) > 0, \\ 0 & \text{if } N(y|x) = 0. \end{cases}$$

2) Confusability graph G on X: adj. matrix

$$(\mathbb{1}+A)_{xx'} = \begin{cases} 1 & \text{if } N(\cdot|x)^T N(\cdot|x') > 0, \\ 0 & \text{if } N(\cdot|x)^T N(\cdot|x') = 0. \end{cases}$$

1) Transition graph Γ: bipartite graph on $X \times Y$ with adjacency matrix

$$\Gamma(y|x) = \begin{cases} 1 & \text{if } N(y|x) > 0, \\ 0 & \text{if } N(y|x) = 0. \end{cases}$$

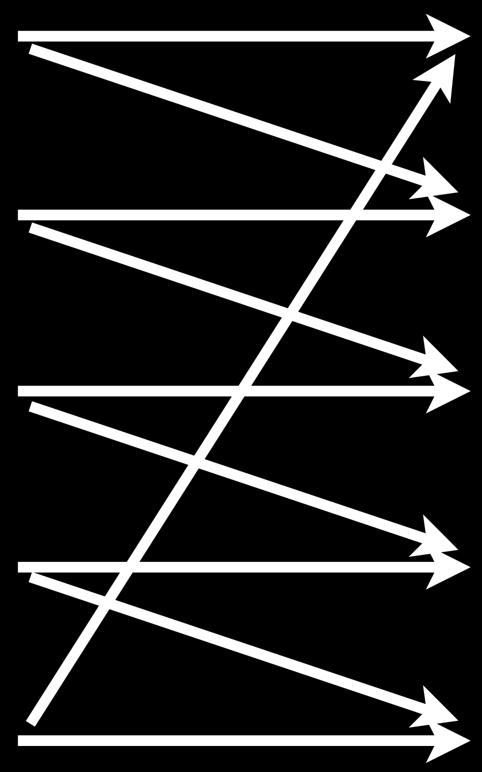
2) Confusability graph G on X: adj. matrix

$$(\mathbb{1}+A)_{xx'} = \begin{cases} 1 & \text{if } N(\cdot|x)^T N(\cdot|x') > 0, \\ 0 & \text{if } N(\cdot|x)^T N(\cdot|x') = 0. \end{cases}$$

Lovász convention:

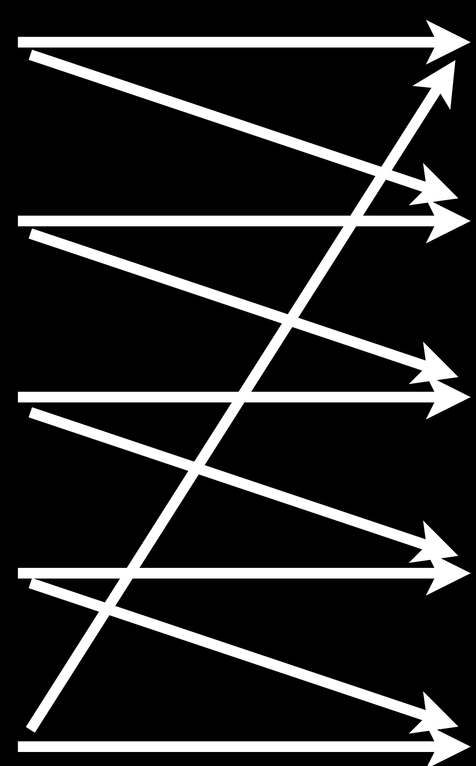
$x \sim x'$ iff $x = x'$ or xx' edge

Γ



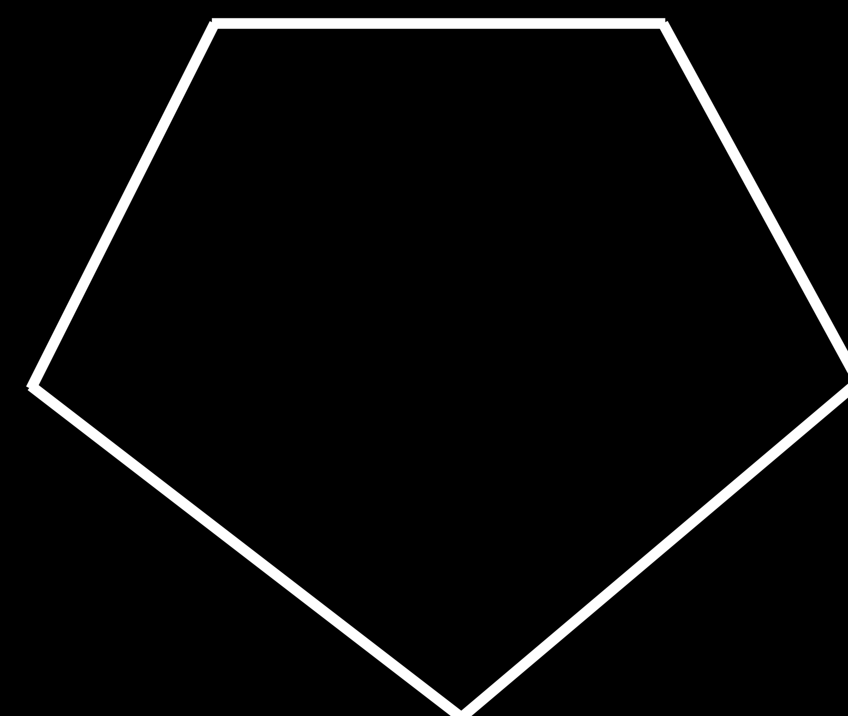
typewriter
channel

Γ



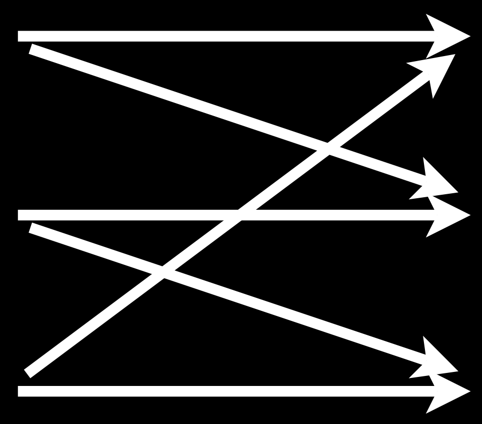
typewriter channel

$G = C_5$

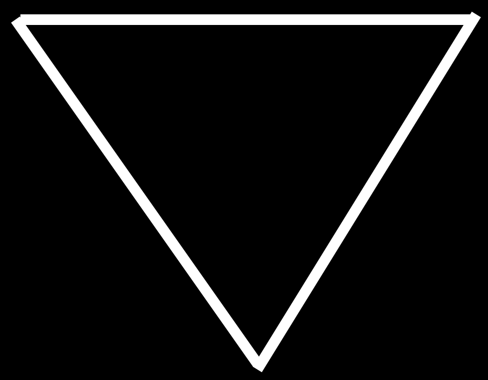


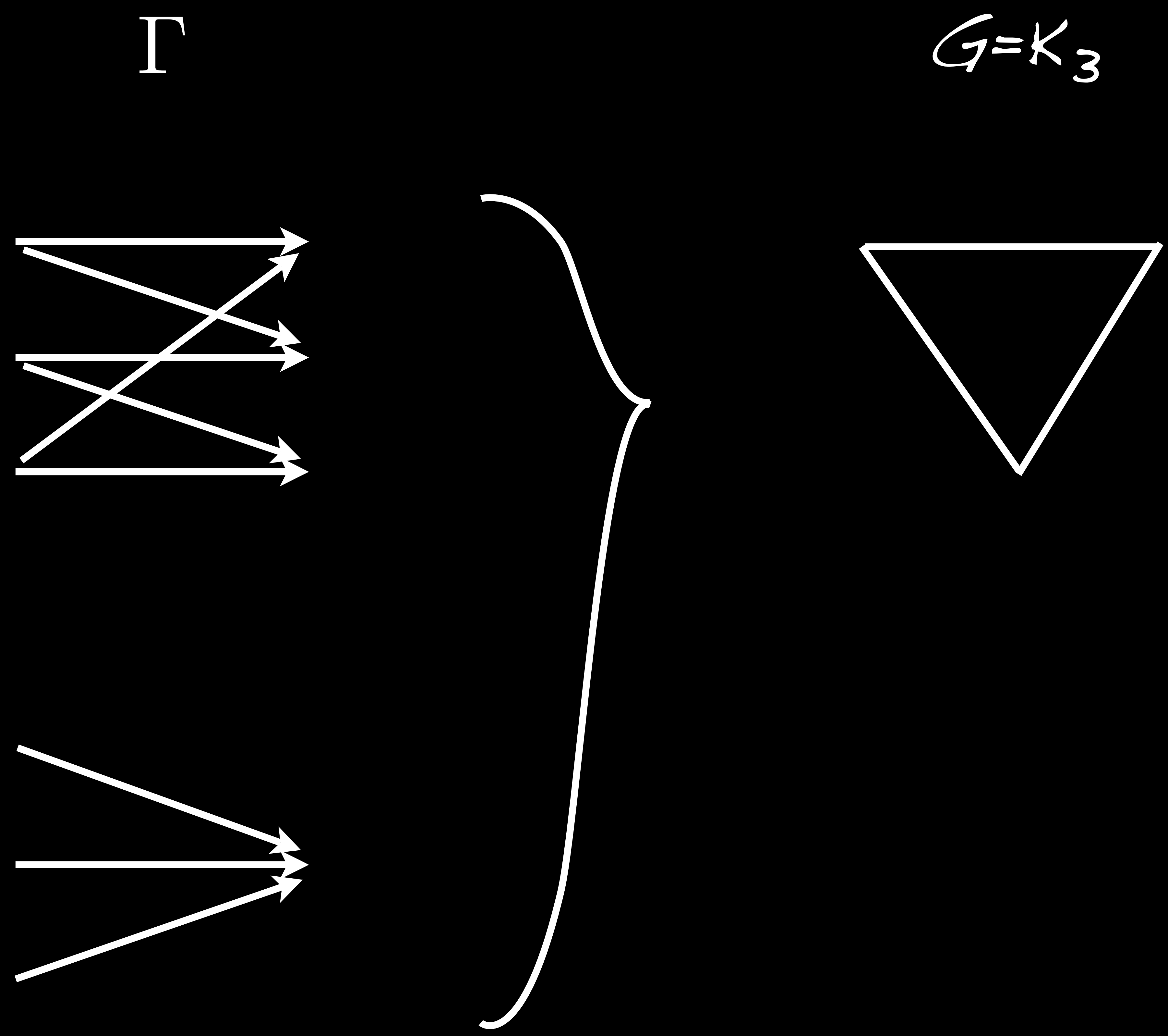
pentagon

Γ



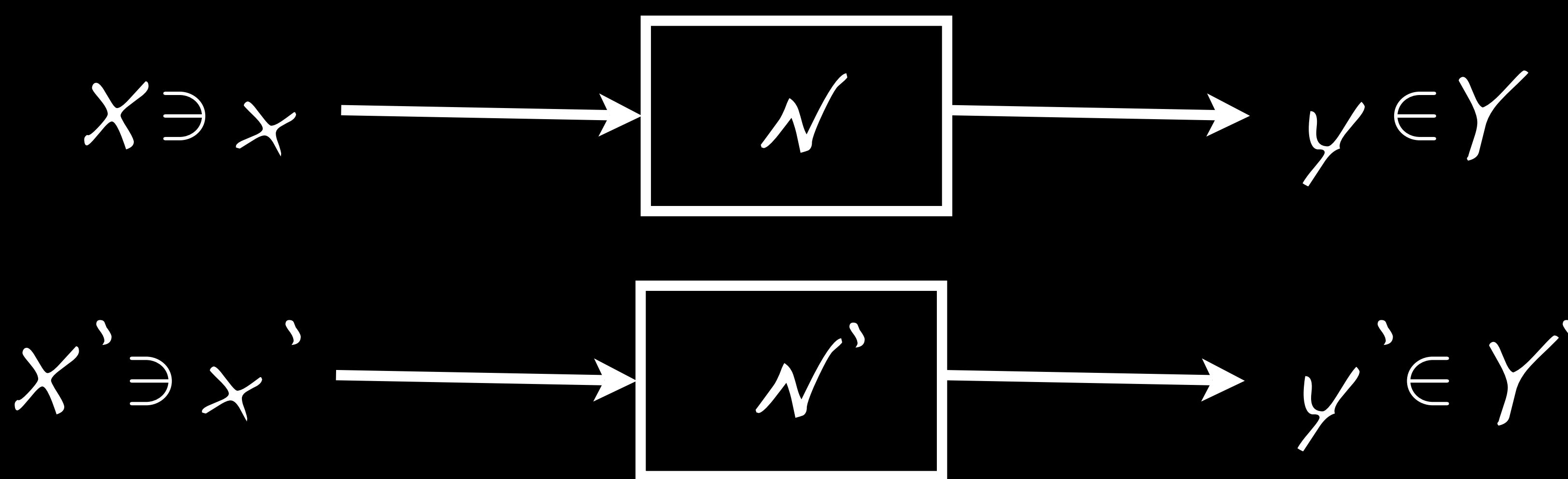
$G = K_3$




Γ
G=K3

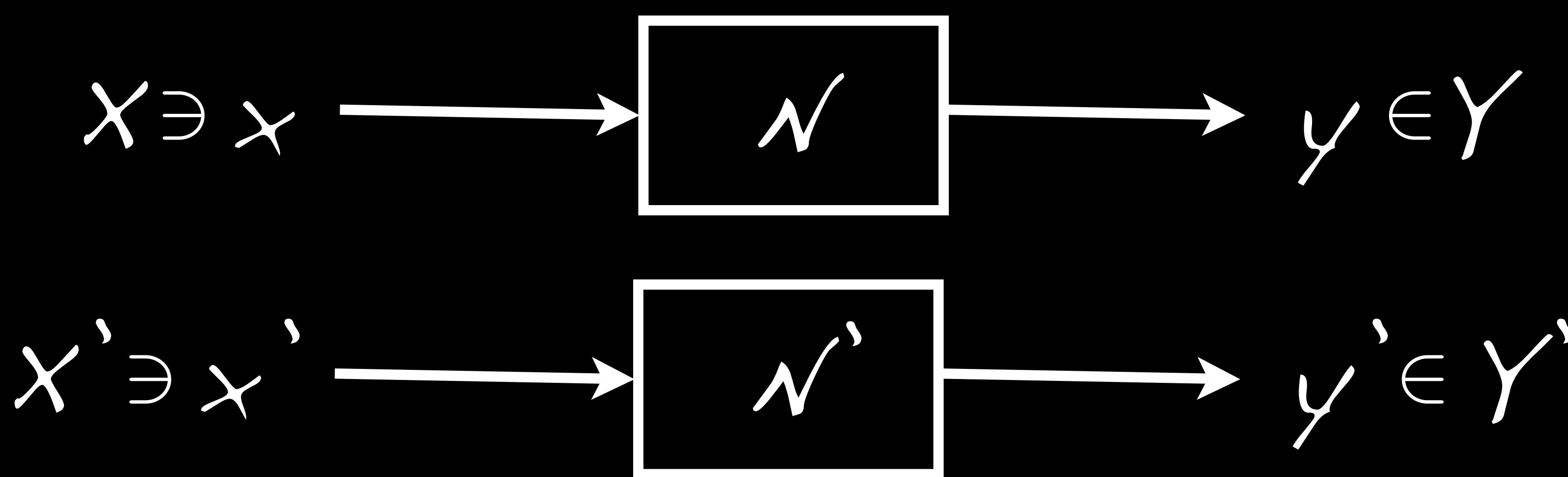
Product channels:

$$N\times N'(yy'|xx') = N(y|x)N'(y'|x')$$



Product channels:

$$N \times N'(yy'|xx') = N(y|x)N'(y'|x')$$



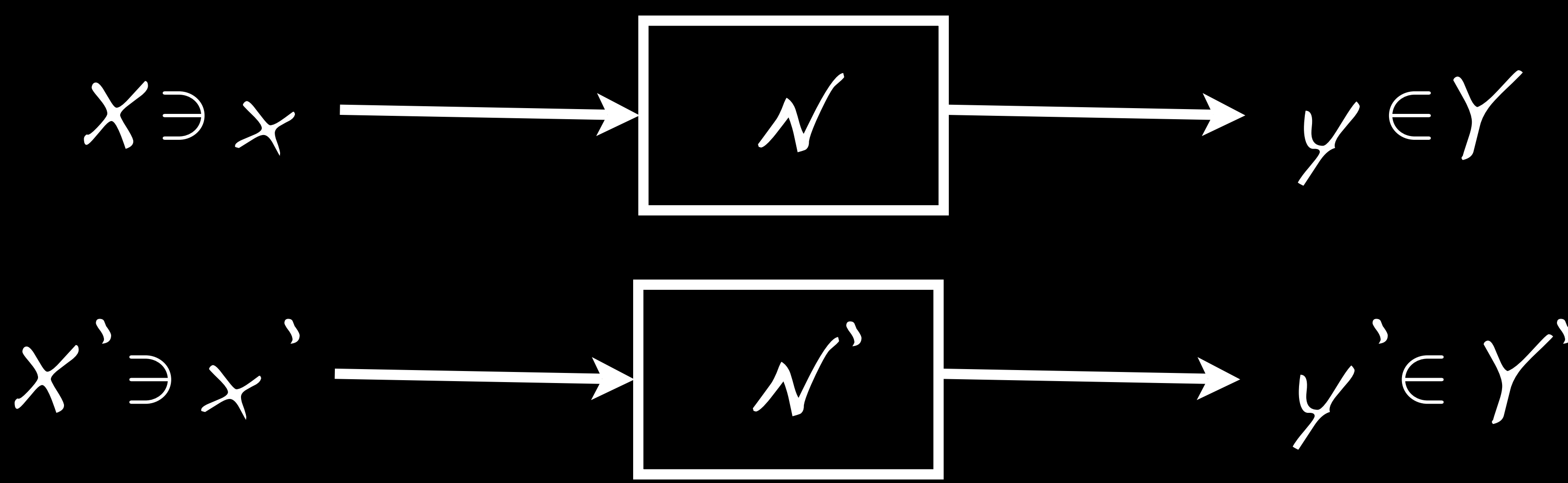
Graphs via Kronecker/tensor product:

$$\Gamma(N \times N') = \Gamma \otimes \Gamma'$$

$$\mathbb{1} + A(N \times N') = (\mathbb{1} + A) \otimes (\mathbb{1} + A')$$

Product channels:

$$N \times N'(yy'|xx') = N(y|x)N'(y'|x')$$



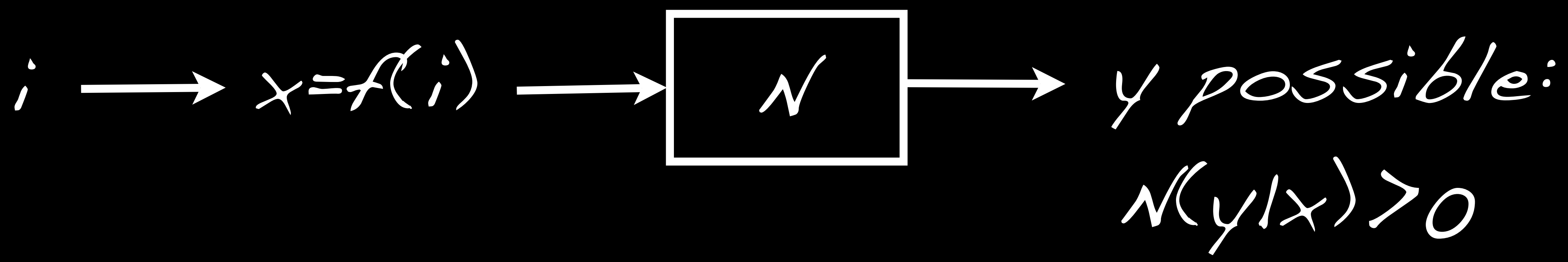
Graphs via Kronecker/tensor product:

$$\Gamma(N \times N') = \Gamma \otimes \Gamma'$$

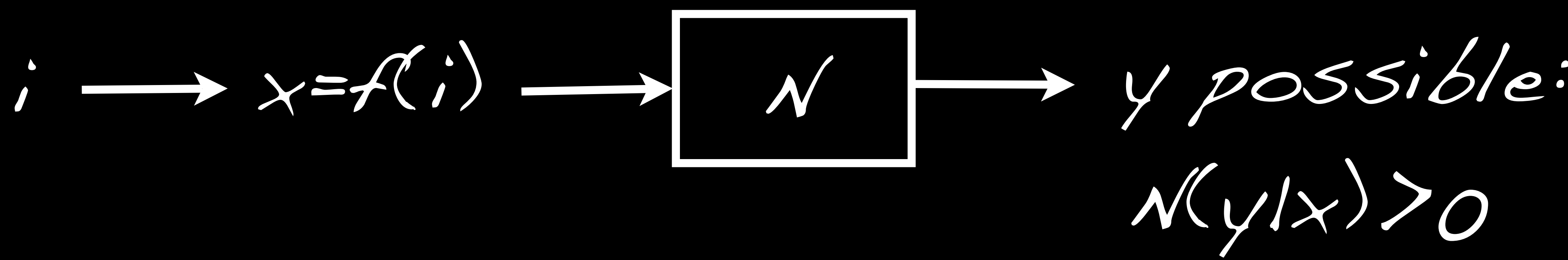
$$\mathbb{1} + A(N \times N') = (\mathbb{1} + A) \otimes (\mathbb{1} + A')$$

Strong graph product $G \times G'$

2. Zero-error transmission



2. Zero-error transmission

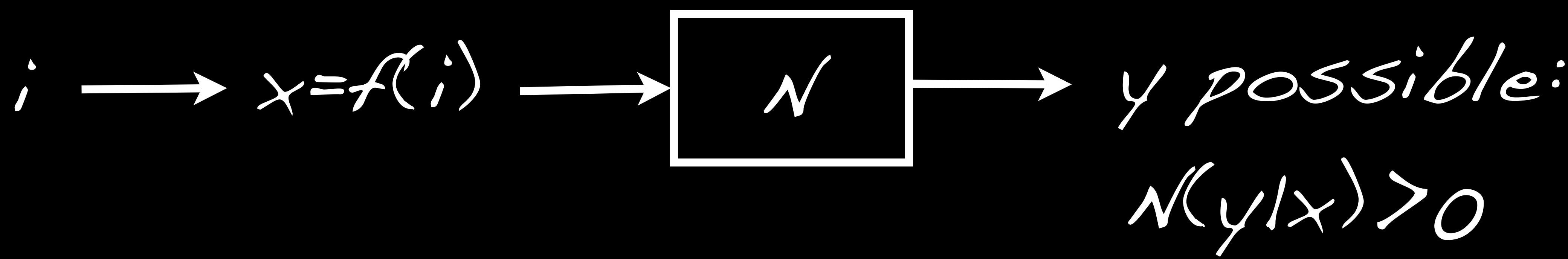


Hence: codebook $\{f(i)\} \subset X$ must be an independent set in G.

Maximum size:

$\alpha(G)$:= independence number of G.

2. Zero-error transmission



Hence: codebook $\{f(i)\} \subset X$ must be an independent set in G.

Maximum size:

$\alpha(G)$:= independence number of G.

Well-known to be NP-complete!

...with assisting resources

Cf. C.E. Shannon [IRE-IT 2(3):8-19, 1956] allowing instantaneous feedback between channel uses.

...with assisting resources

Cf. C.E. Shannon [IRE-IT 2(3):8-19, 1956] allowing instantaneous feedback between channel uses.

Here study impact of correlations: clearly shared randomness doesn't change $\alpha(G)$; consider interactive devices.

...with assisting resources

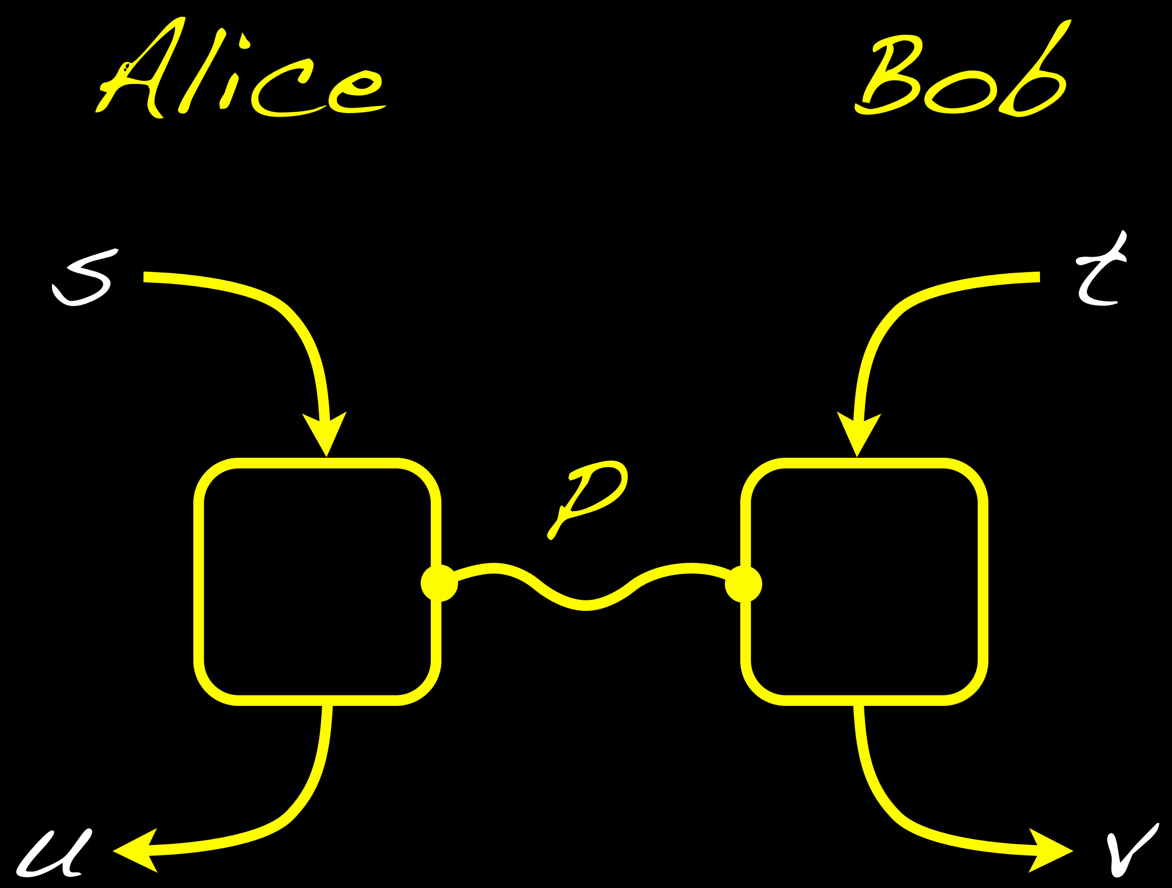
Cf. C.E. Shannon [IRE-IT 2(3):8-19, 1956] allowing instantaneous feedback between channel uses.

Here study impact of correlations: clearly shared randomness doesn't change $\alpha(G)$; consider interactive devices.

Mathematically: bidirectional channels $P(uv|st)$ – that do not allow signalling...

Cf. C.E. Shannon [Proc. 4th Berkeley Symp., 1961] and J. Barrett [PRA 75:032304, 2007].

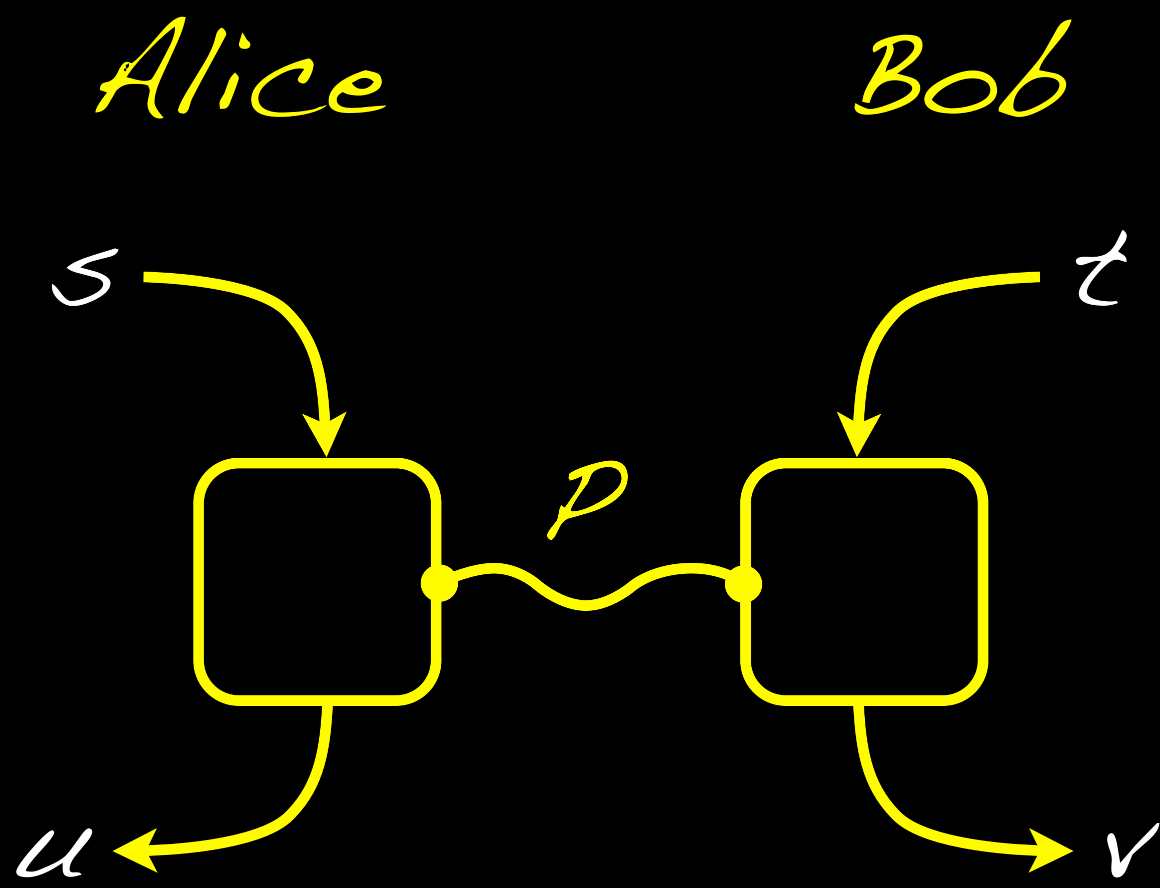
...with assisting resources



Clearly $P(uv|st) \geq 0$,

and $\sum_{uv} P(uv|st) = 1$.

...with assisting resources



Clearly $P(uv|st) \geq 0$,

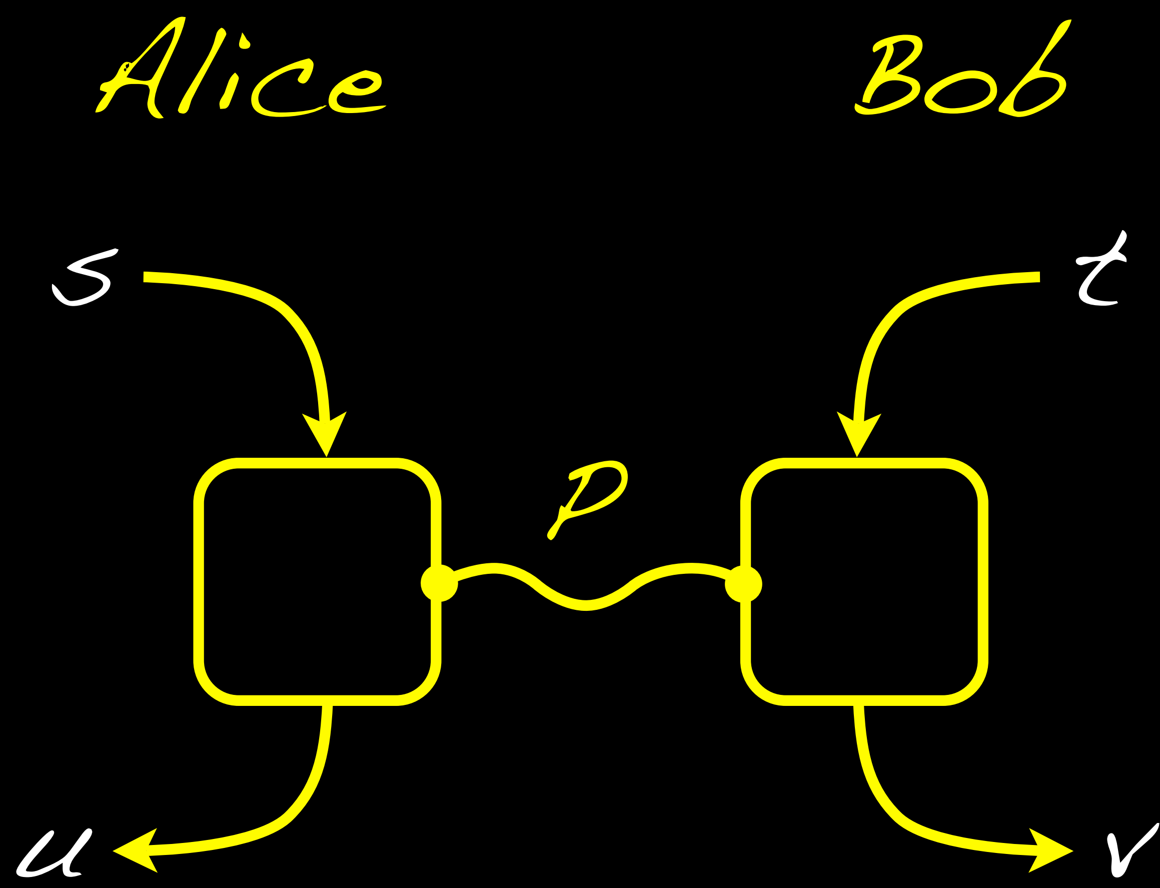
and $\sum_{uv} P(uv|st) = 1$.

No-signalling means

$$\sum_{u} P(uv|st) = P(v|t),$$

$$\sum_{v} P(uv|st) = P(u|s).$$

...with assisting resources



Clearly $P(uv|st) \geq 0$,

and $\sum_{uv} P(uv|st) = 1$.

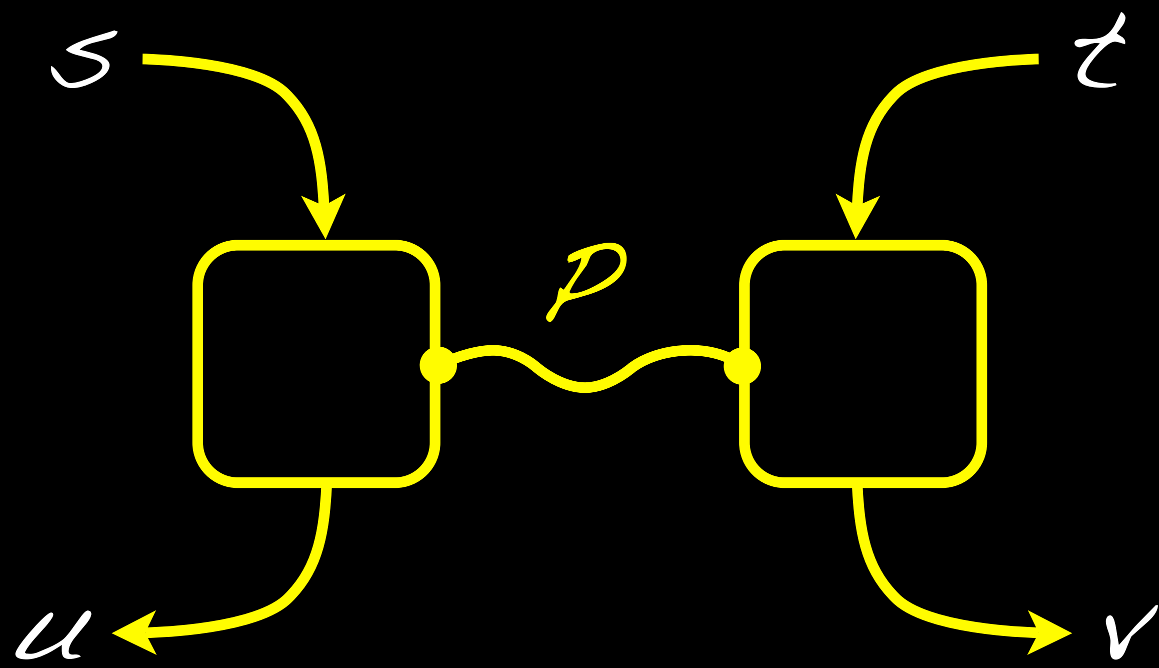
No-signalling means

$$\sum_{u} P(uv|st) = P(v|t),$$

$$\sum_{v} P(uv|st) = P(u|s).$$

Set NS of these: convex set, in fact a polytope, of interesting structure.

...with assisting resources



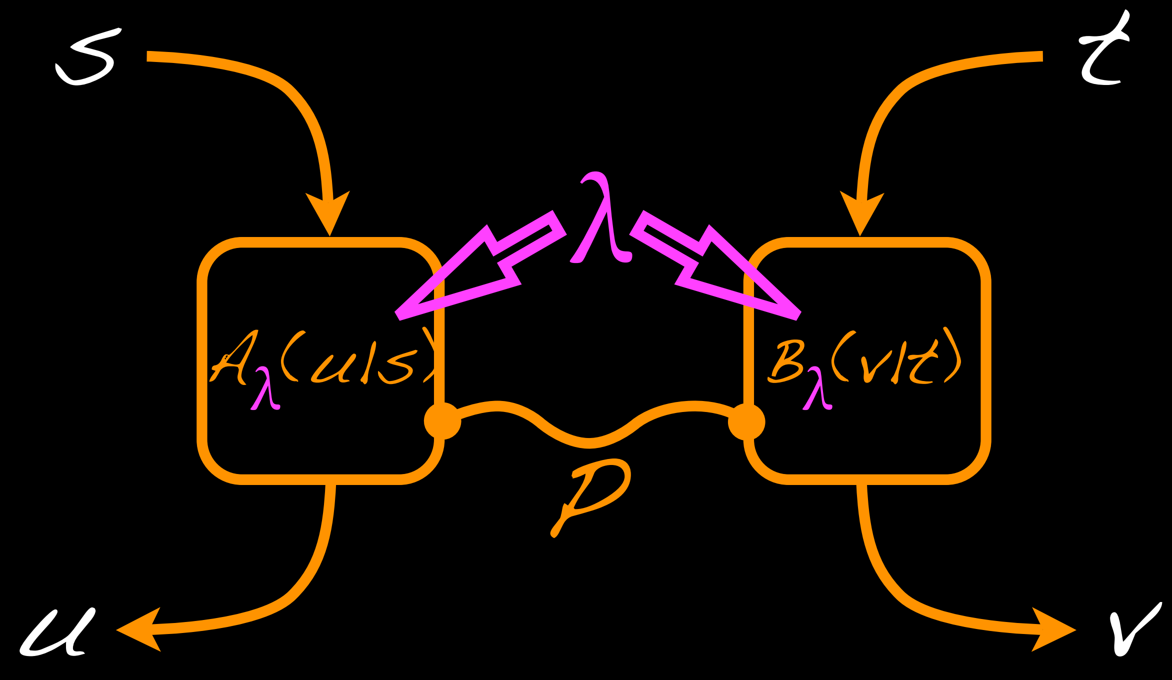
No-signalling means

$$\sum_u P(uv|st) = P(v|t),$$

$$\sum_v P(uv|st) = P(u|s).$$

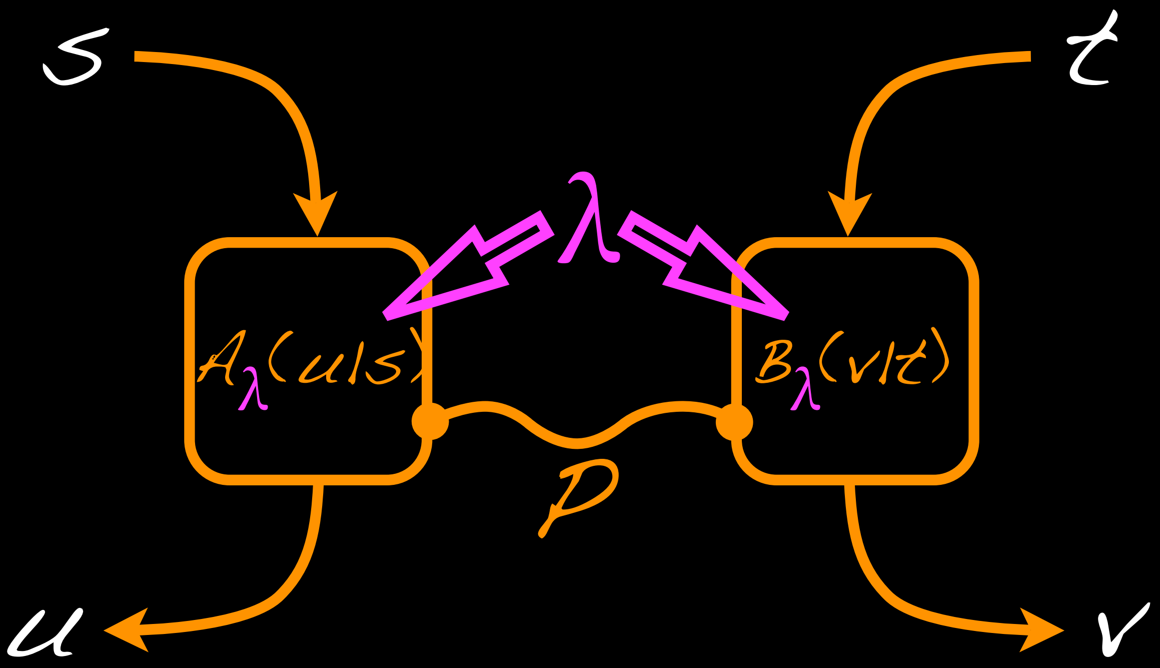
Although formally a channel with two simultaneous inputs, the no-signalling condition ensures that Alice can use the "box" without waiting. Bob is left with a conditional $P_{us}(v|t) = P(uv|st)/P(u|s)$.

Special subsets – "local" correlations owed to shared randomness and local operations:



$$LHV := \text{conv}\{ A(u|s)B(v|t) : A, B \text{ functions}\}$$

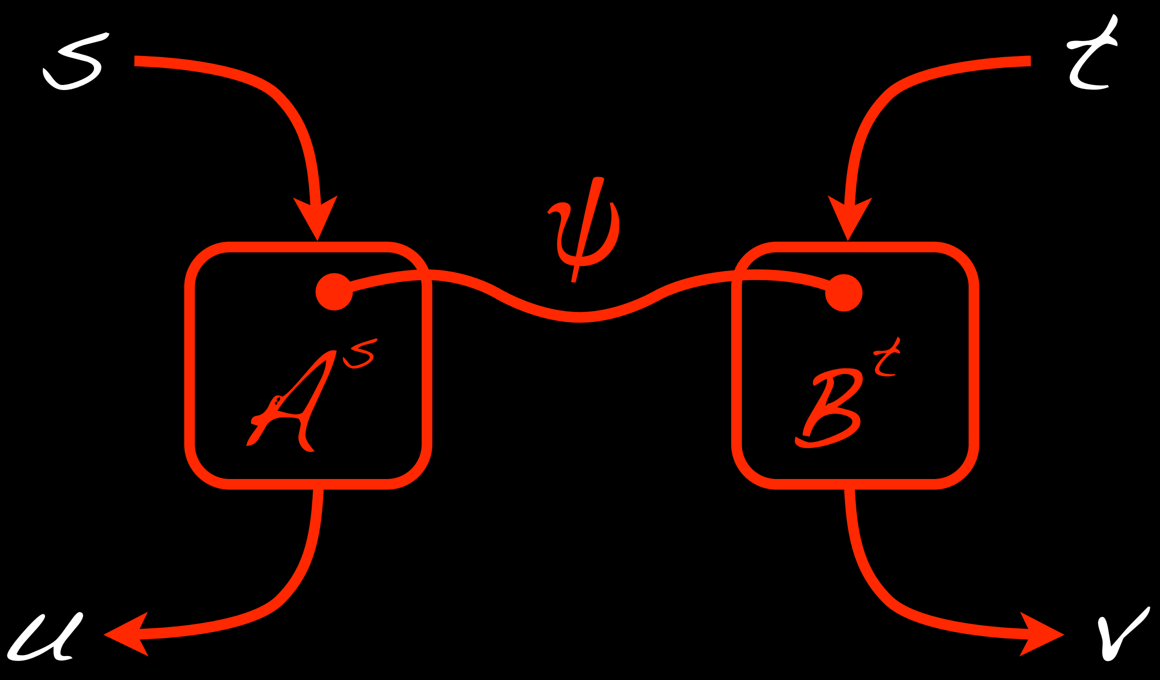
Special subsets – "local" correlations owed to shared randomness and local operations:



$$LHV := \text{conv}\{ A(u|s)B(v|t) : A, B \text{ functions} \}$$

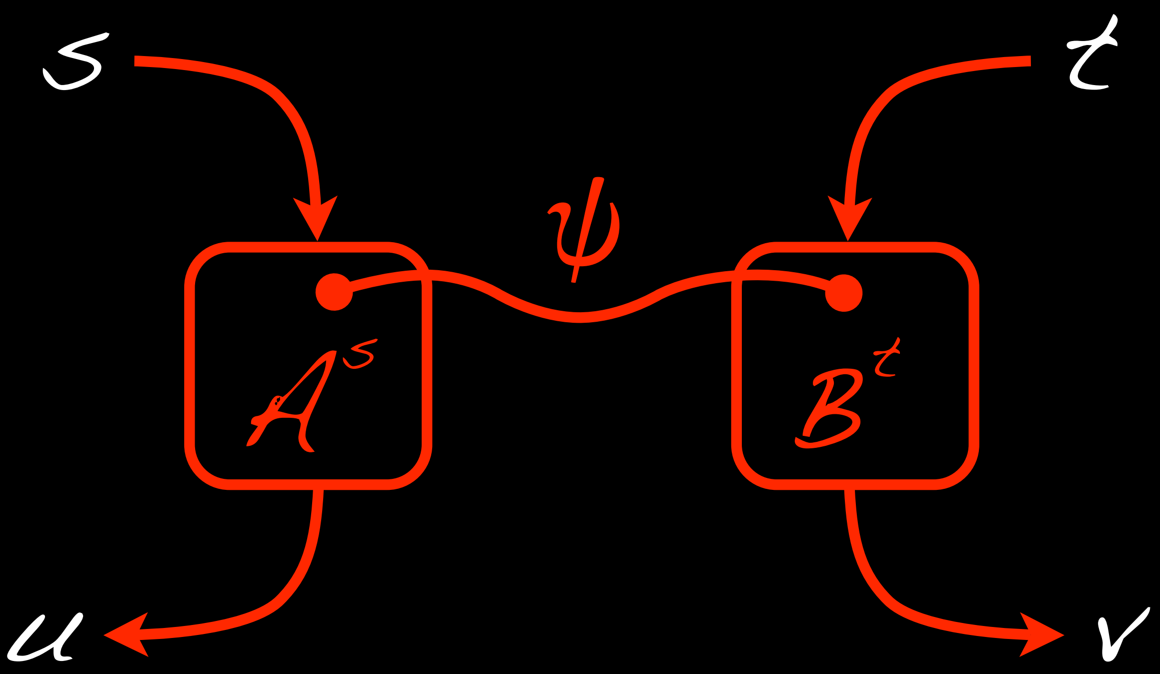
$$LHV \subset NS$$

Special subsets - quantum correlations owed to entanglement and local measurements:



$Q = \{P(uv|st) = Tr\ \psi(A^s_u \otimes B^t_v) : \psi$ state and A^s, B^t local observables$\}$

Special subsets - quantum correlations owed to entanglement and local measurements:



$$Q = \{ P(uv|st) = \mathrm{Tr}\, \psi (A^s_u \otimes B^t_v) : \psi \text{ state and } A^s, B^t \text{ local observables} \}$$

$$LHV \subset Q \subset NS$$

Inclusions of the classes are strict:

$$LHV = \mathrm{conv}\{A(u|s)B(v|t)\}$$

$$\cap$$

$$Q = \{P(uv|st) = \mathrm{Tr}\, \psi(A_u^s \otimes B_v^t)\}$$

$$\cap$$

$$NS = \{P(uv|st) \text{ no-signalling}\}$$

Inclusions of the classes are strict:

$$LHV = \mathrm{conv}\,\{A(u|s)B(v|t)\}$$

$$\cap$$

$$Q \;=\; \{P(uv|st) = \mathrm{Tr}\,\psi(A_u^s \otimes B_v^t)\}$$

$$\cap$$

$$NS \;=\; \{P(uv|st) \text{ no-signalling}\}$$

In fact, even for $s,t,u,v \in \{0,1\}$.

Consider uniform s,t and maximize

$\Pr\{u \oplus v = st\}$ over each class...

Bell & Tsirelson inequalities:

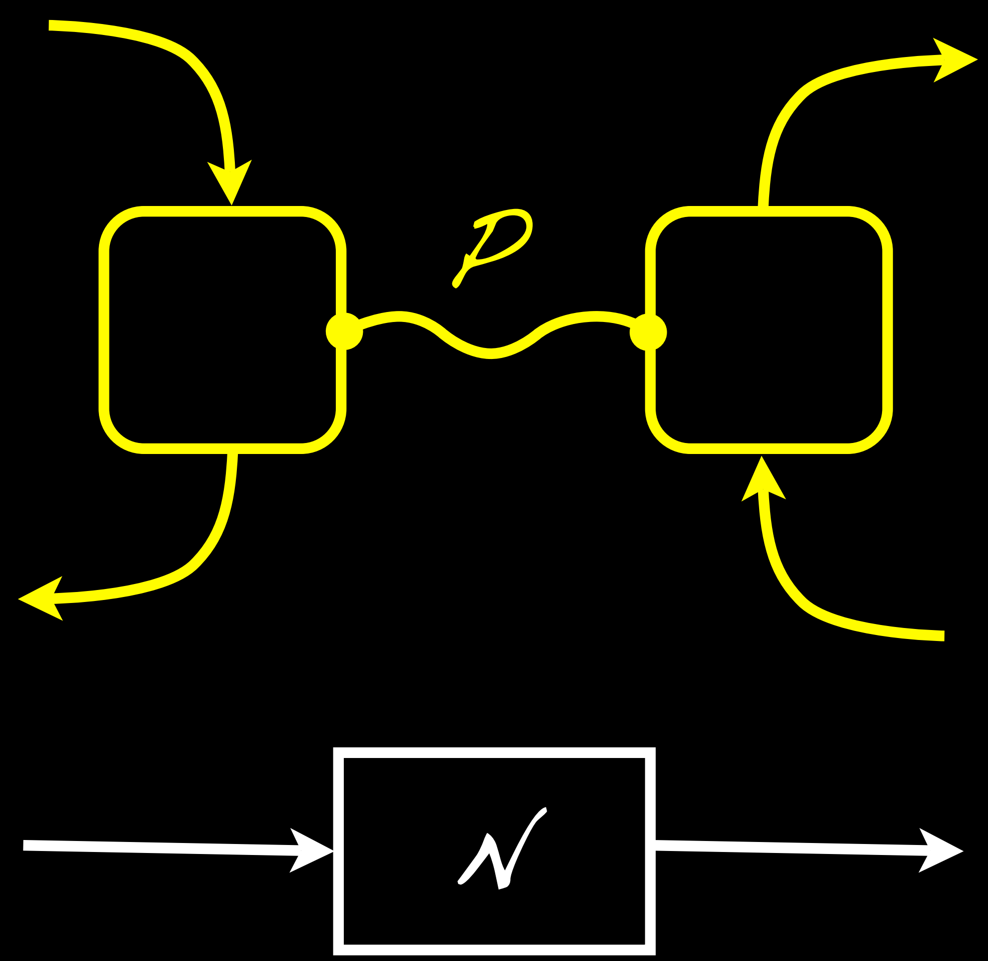
$P \in LHV$: $\Pr\{u \oplus v = st\} \leq 3/4$

$P \in Q$: $\Pr\{u \oplus v = st\} \leq \cos^2 \pi/8 \approx 0.85$

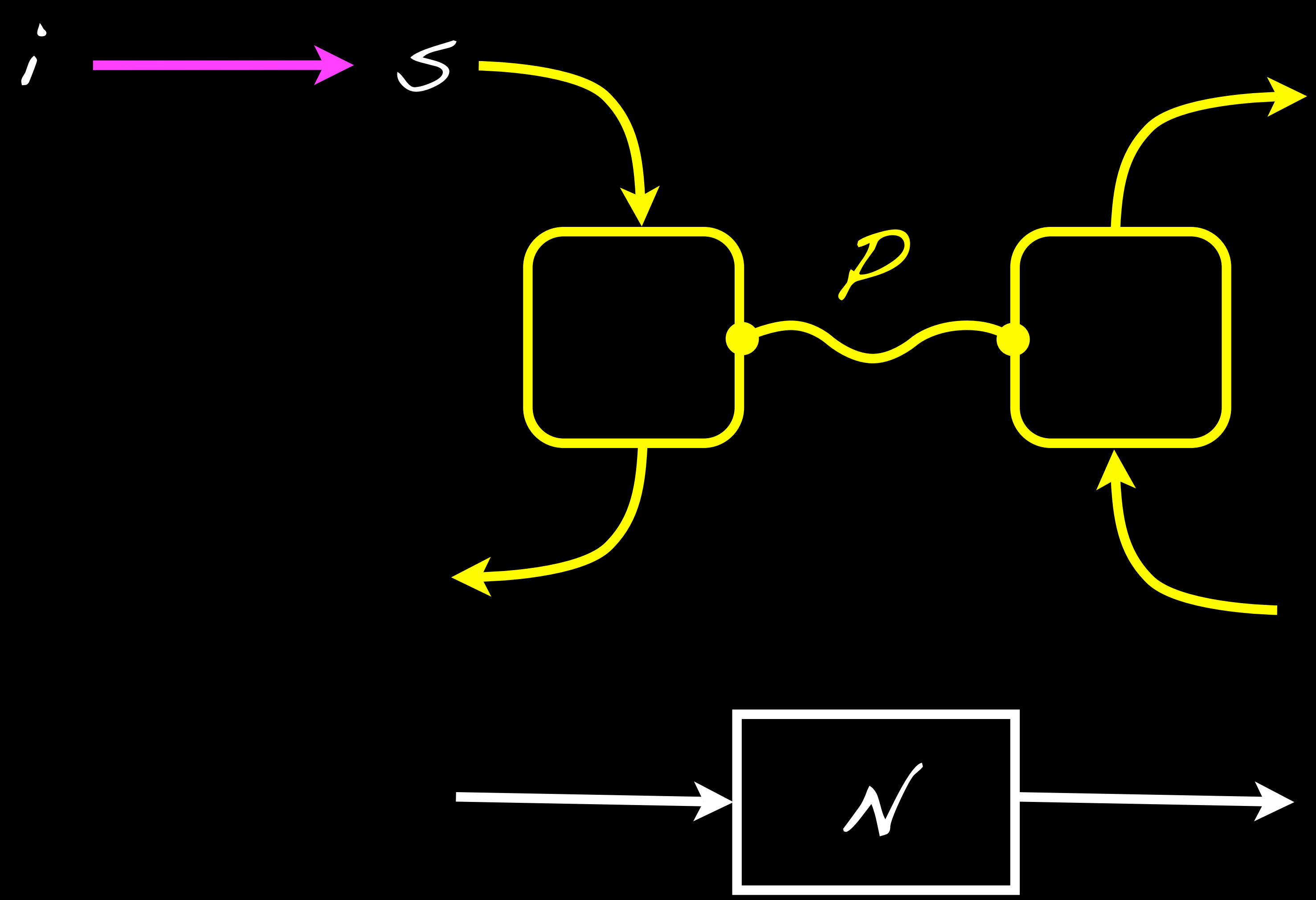
$P \in NS$: $\Pr\{u \oplus v = st\} \leq 1$

[Cf. review by N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani and S. Wehner, arXiv:1303.2849].

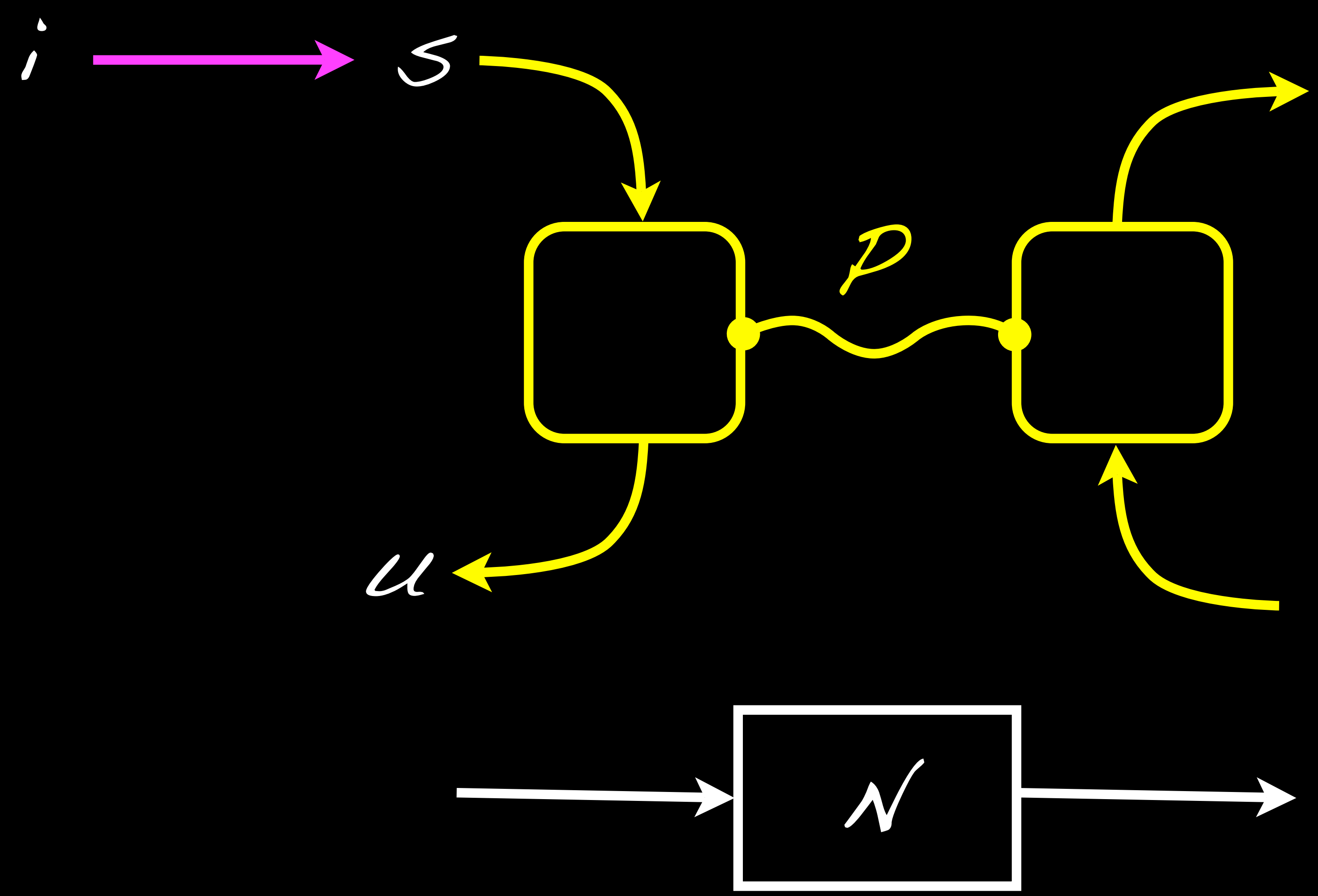
How to use a channel $\mathcal{N}$ in conjunction with a given no-signalling correlation $\mathcal{P}$:



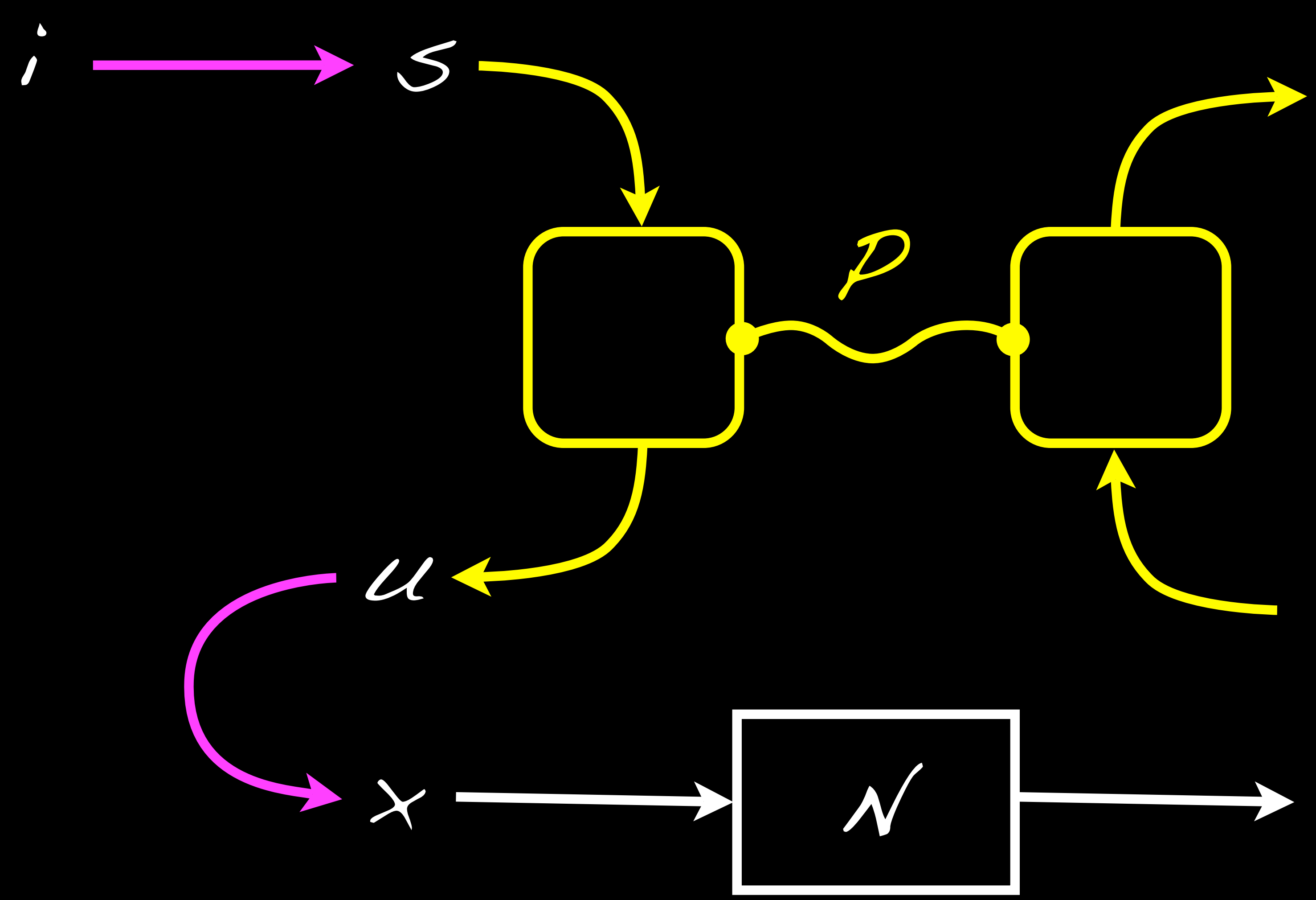
How to use a channel $\mathcal{N}$ in conjunction with a given no-signalling correlation $\mathcal{P}$:



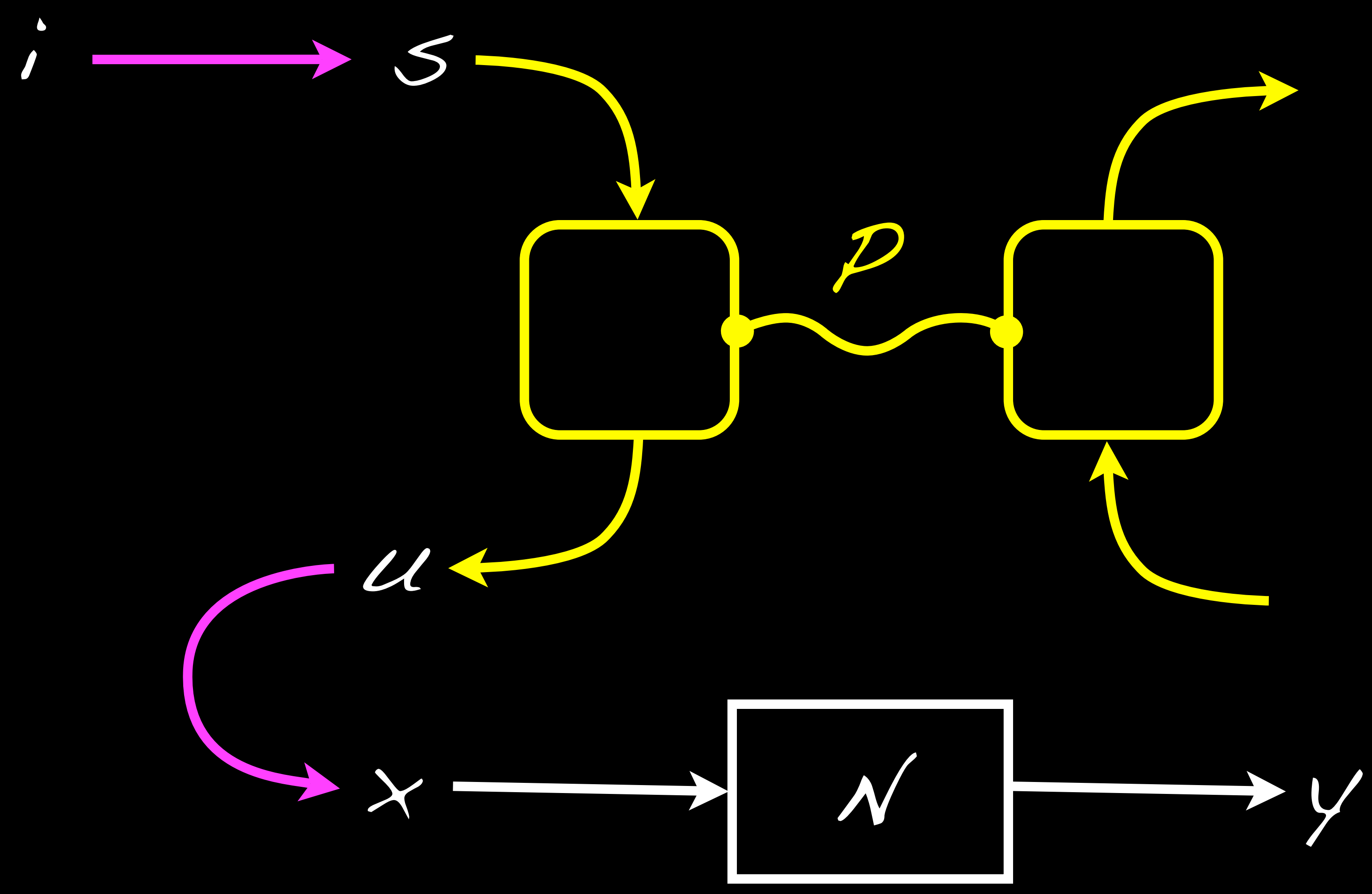
How to use a channel $\mathcal{N}$ in conjunction with a given no-signalling correlation P:



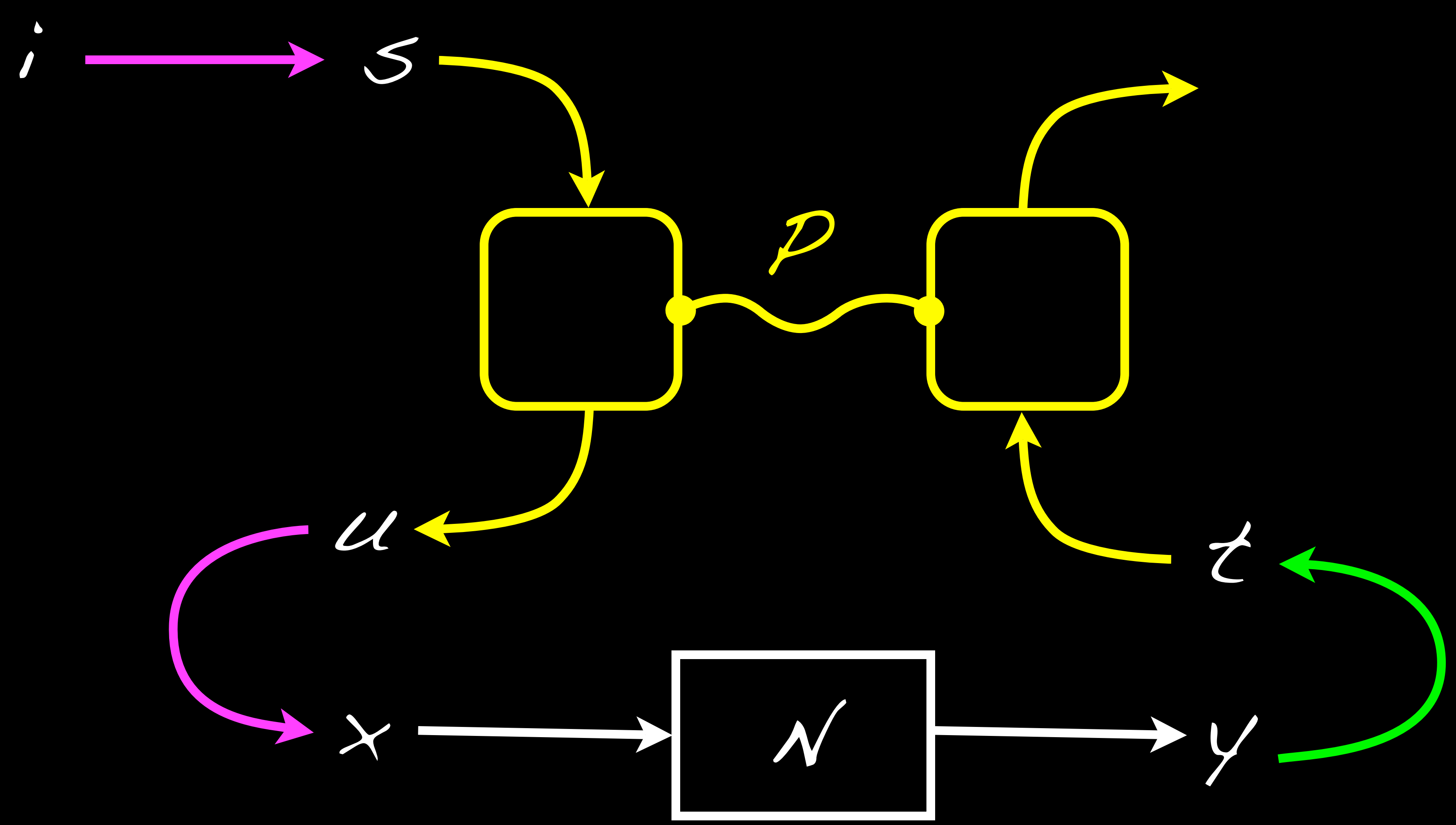
How to use a channel N in conjunction with a given no-signalling correlation P:



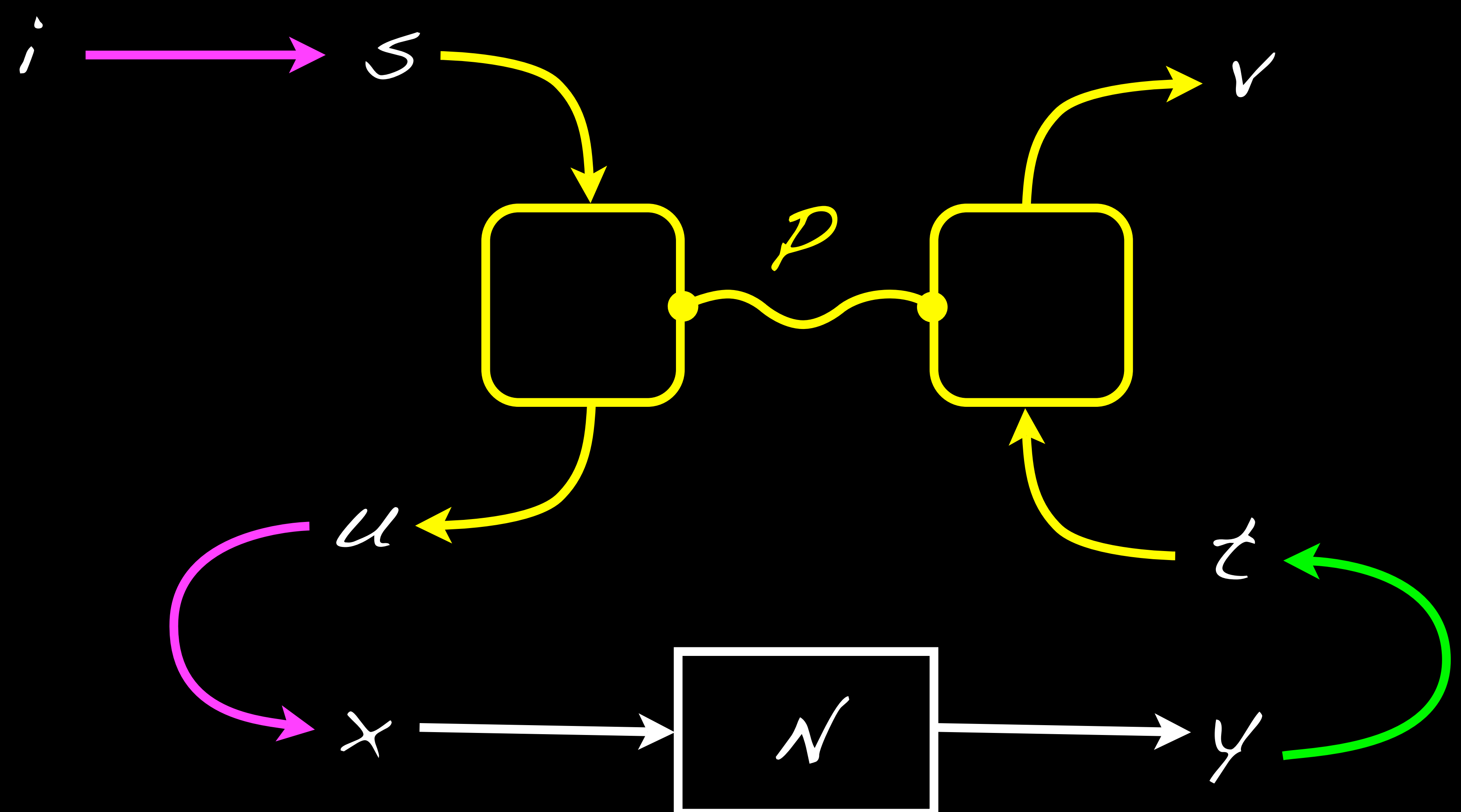
How to use a channel N in conjunction with a given no-signalling correlation P:



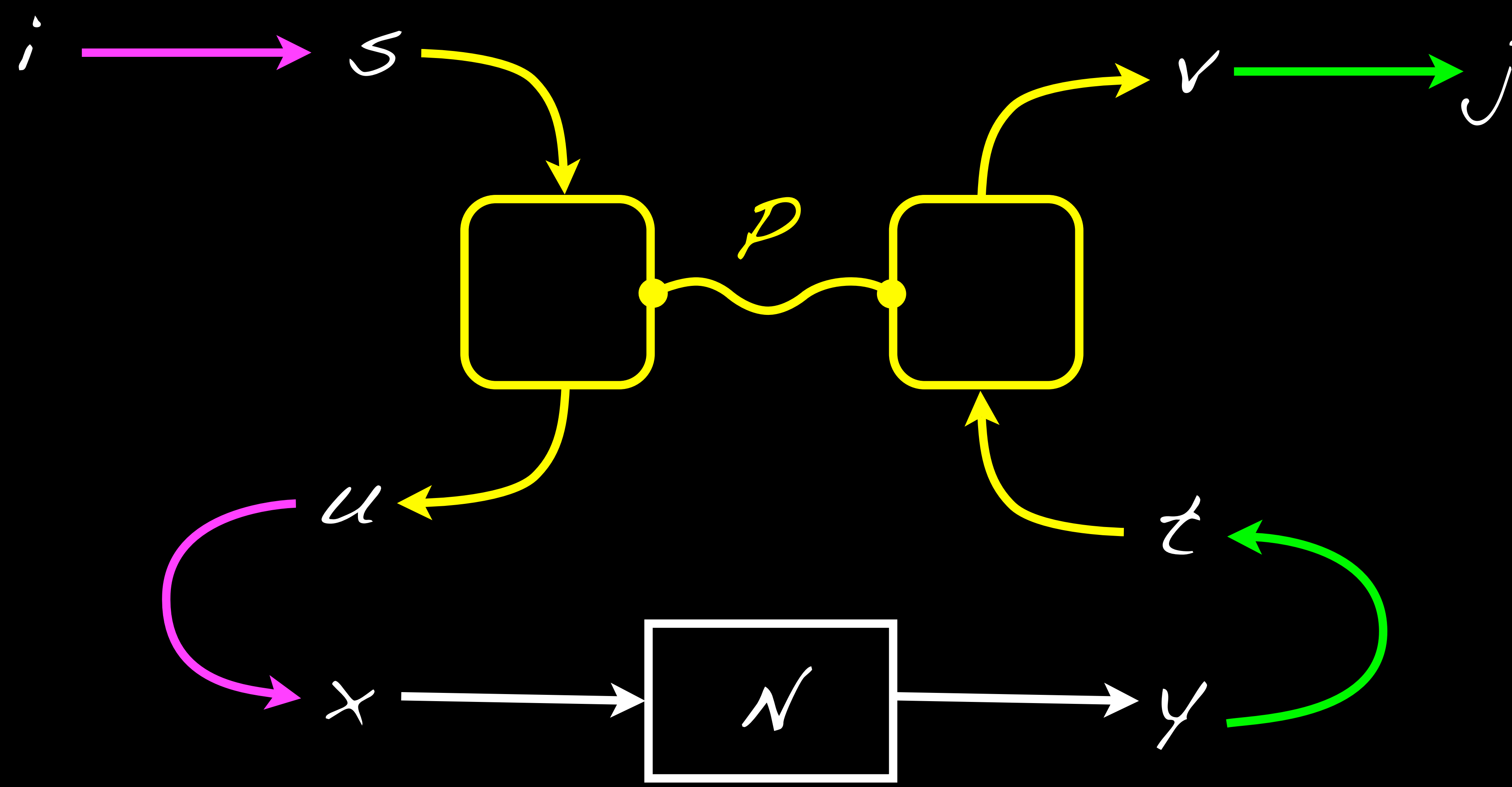
How to use a channel N in conjunction with a given no-signalling correlation P:



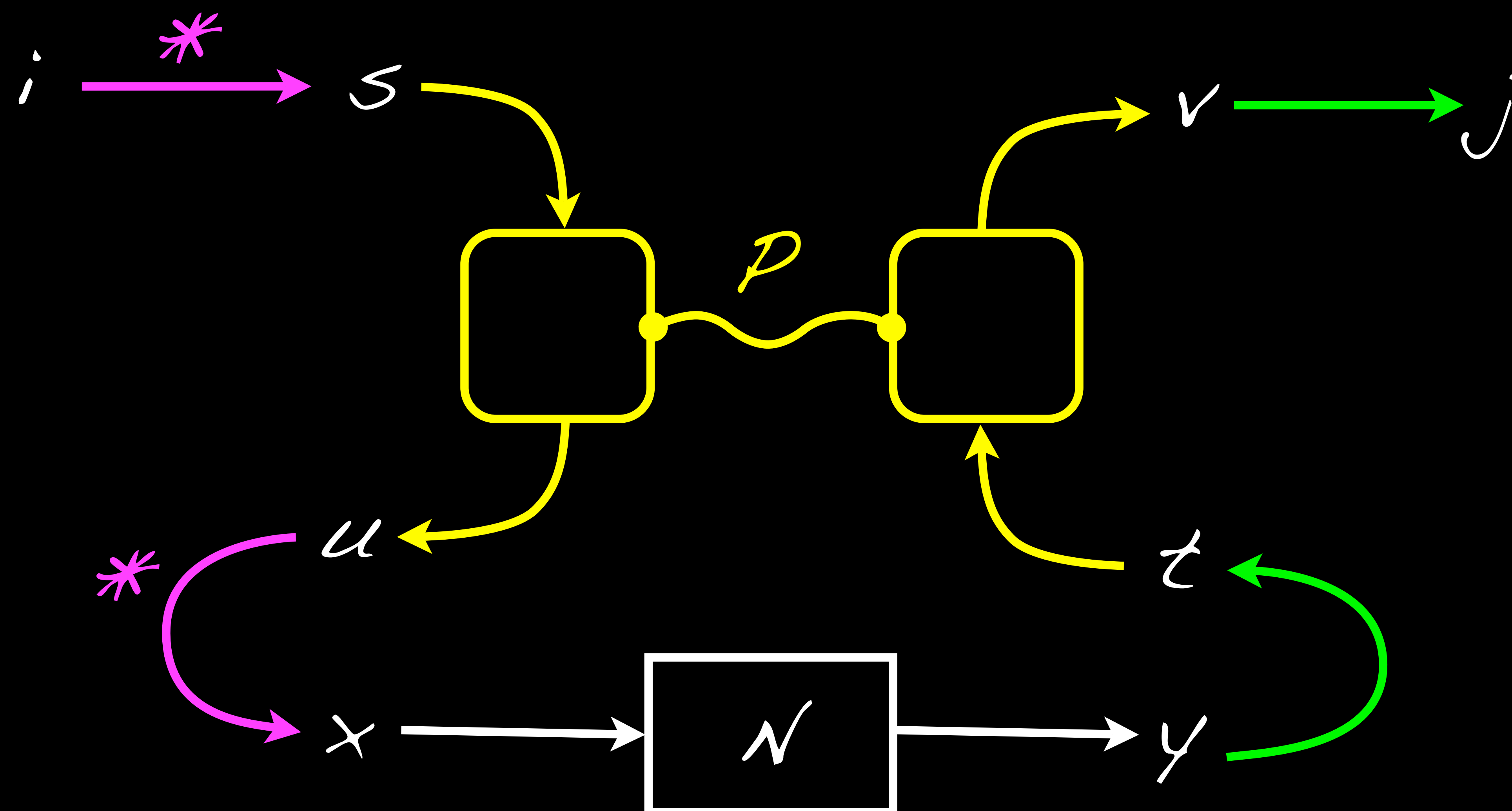
How to use a channel N in conjunction with a given no-signalling correlation P:



How to use a channel $\mathcal{N}$ in conjunction with a given no-signalling correlation P:

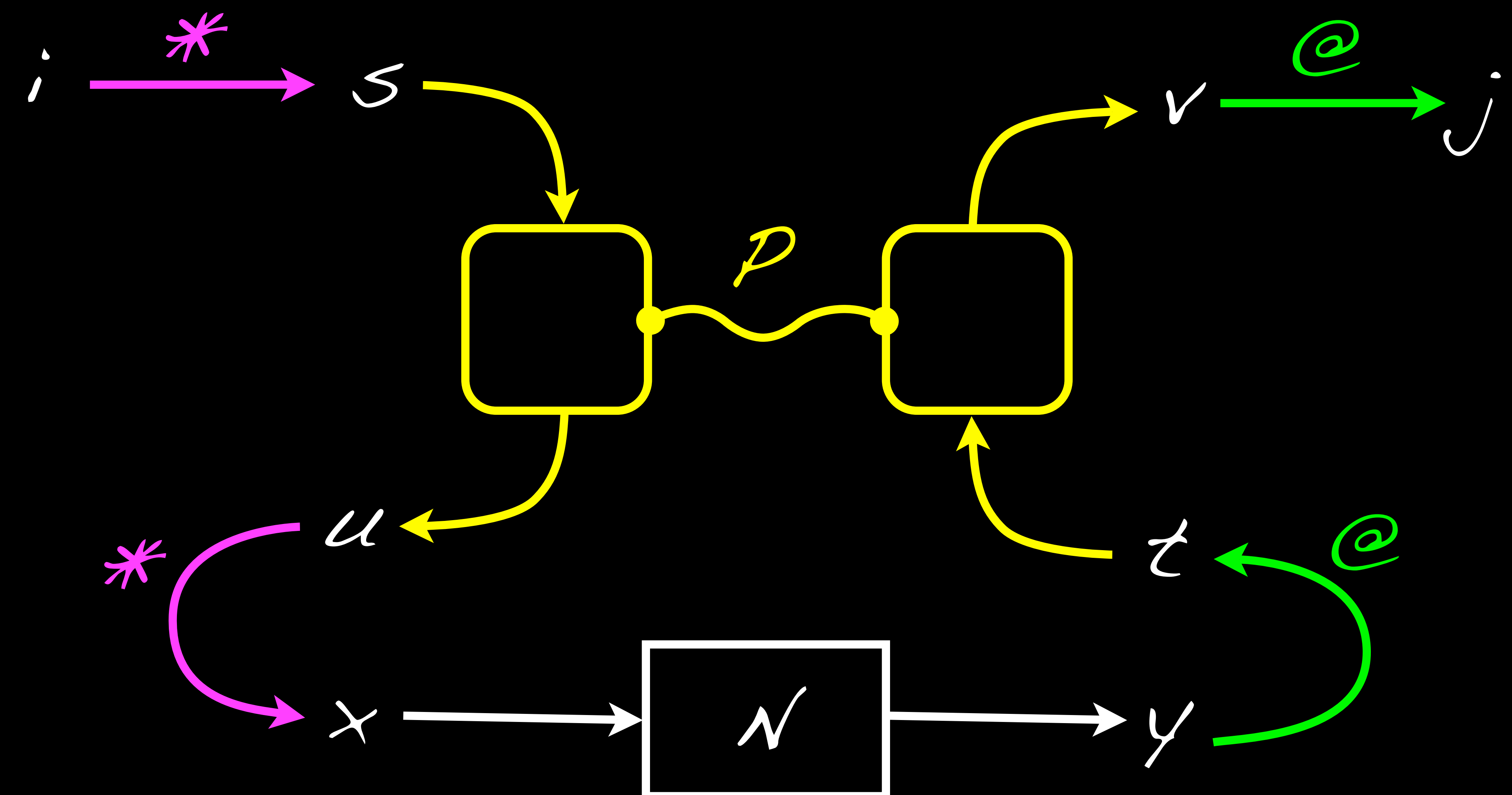


How to use a channel N in conjunction with a given no-signalling correlation P:



* Alice's mappings

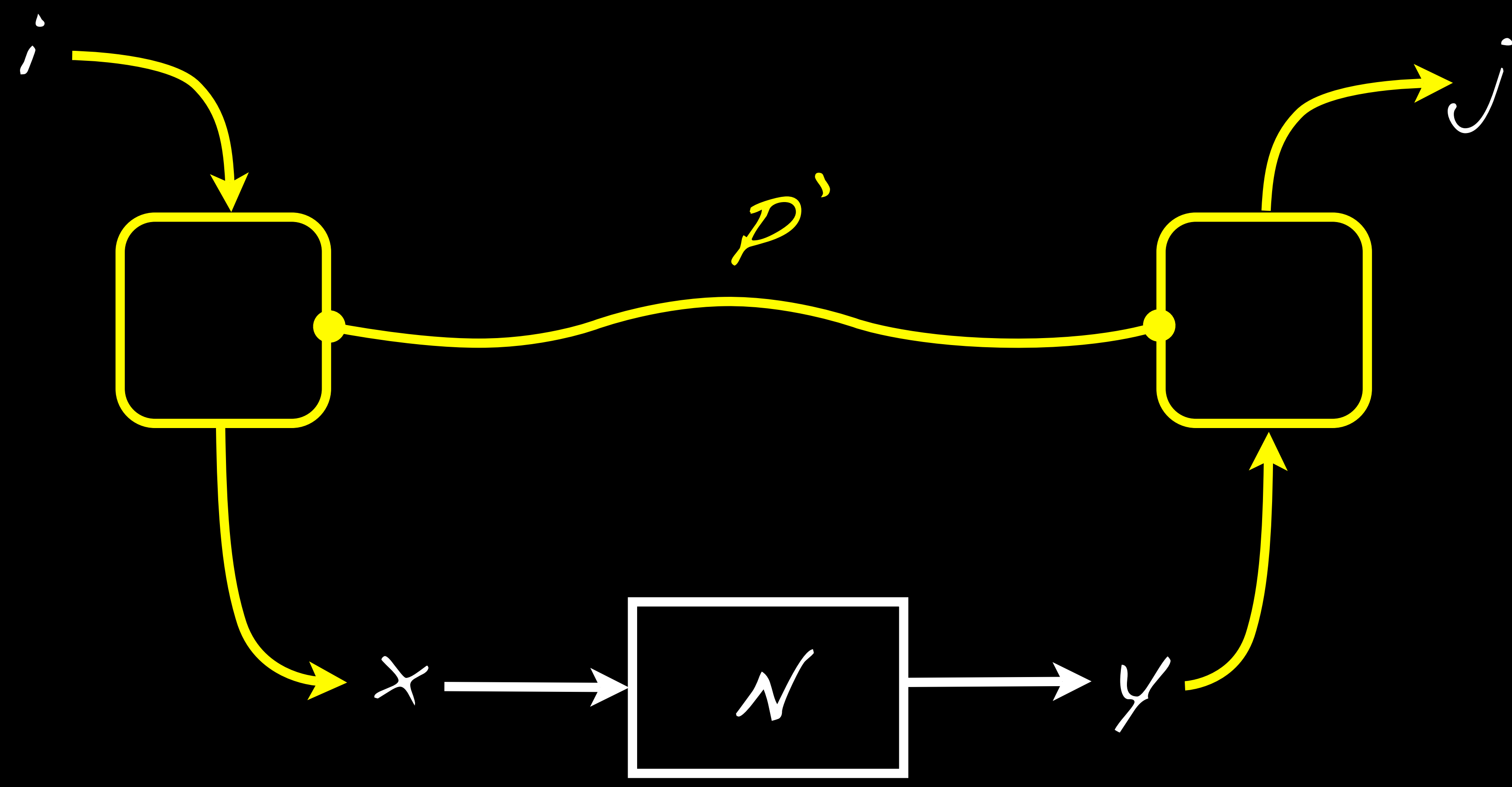
How to use a channel N in conjunction with a given no-signalling correlation P:



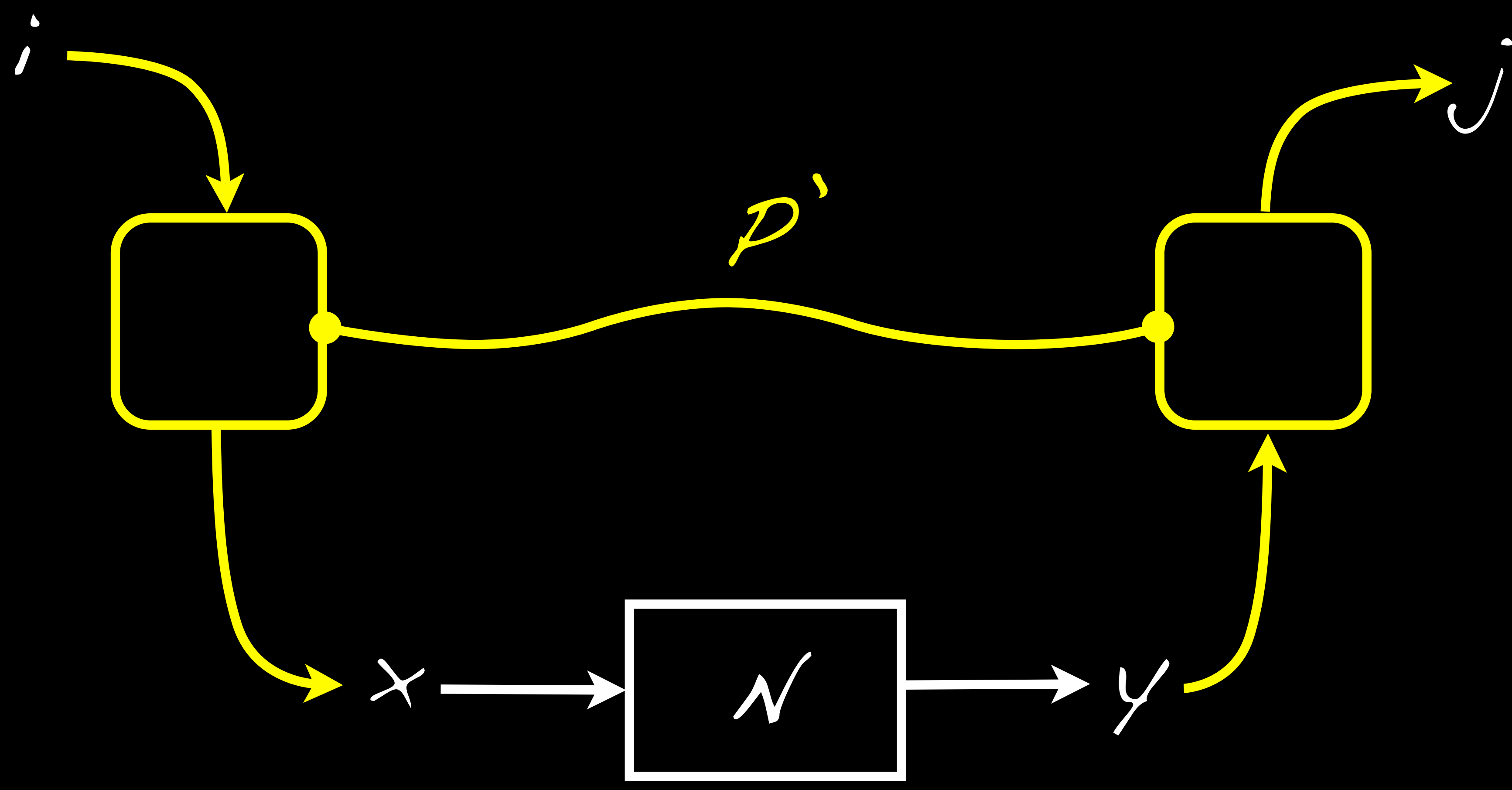
* Alice's mappings

@ Bob's mappings

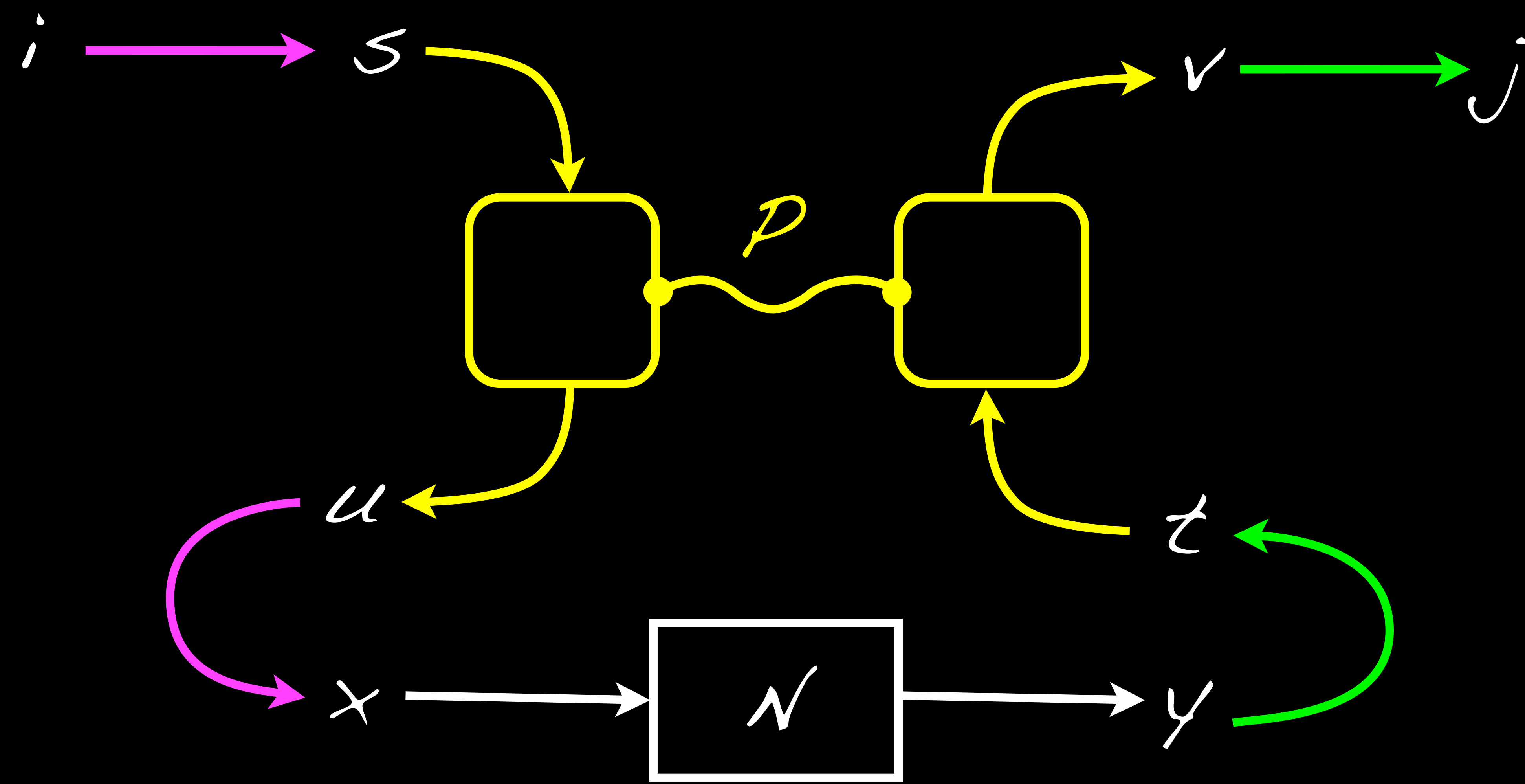
By absorbing the local operations into P:



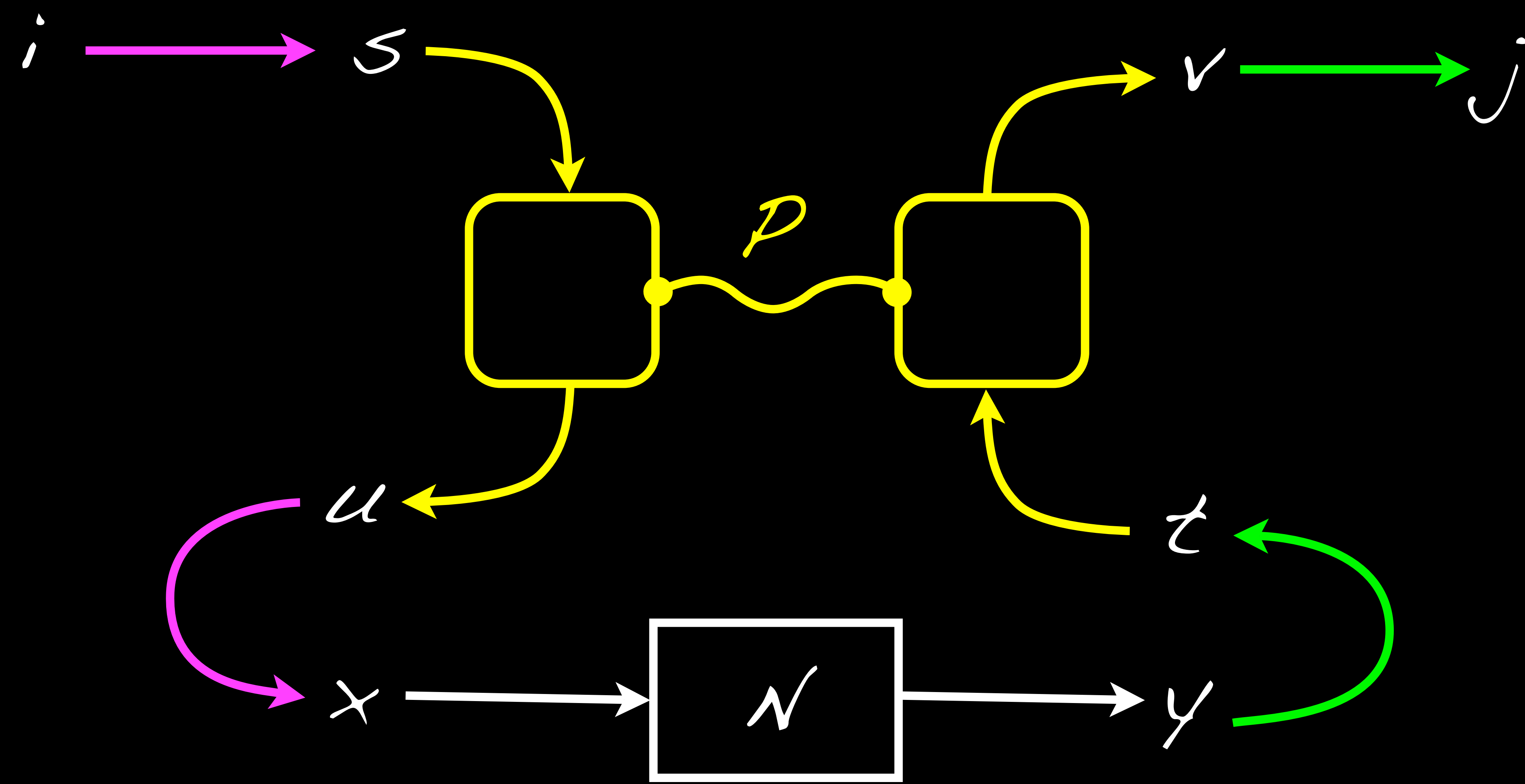
By absorbing the local operations into P:



P'(xj|iy) - is in LHV / Q / NS if P was in the respective class!

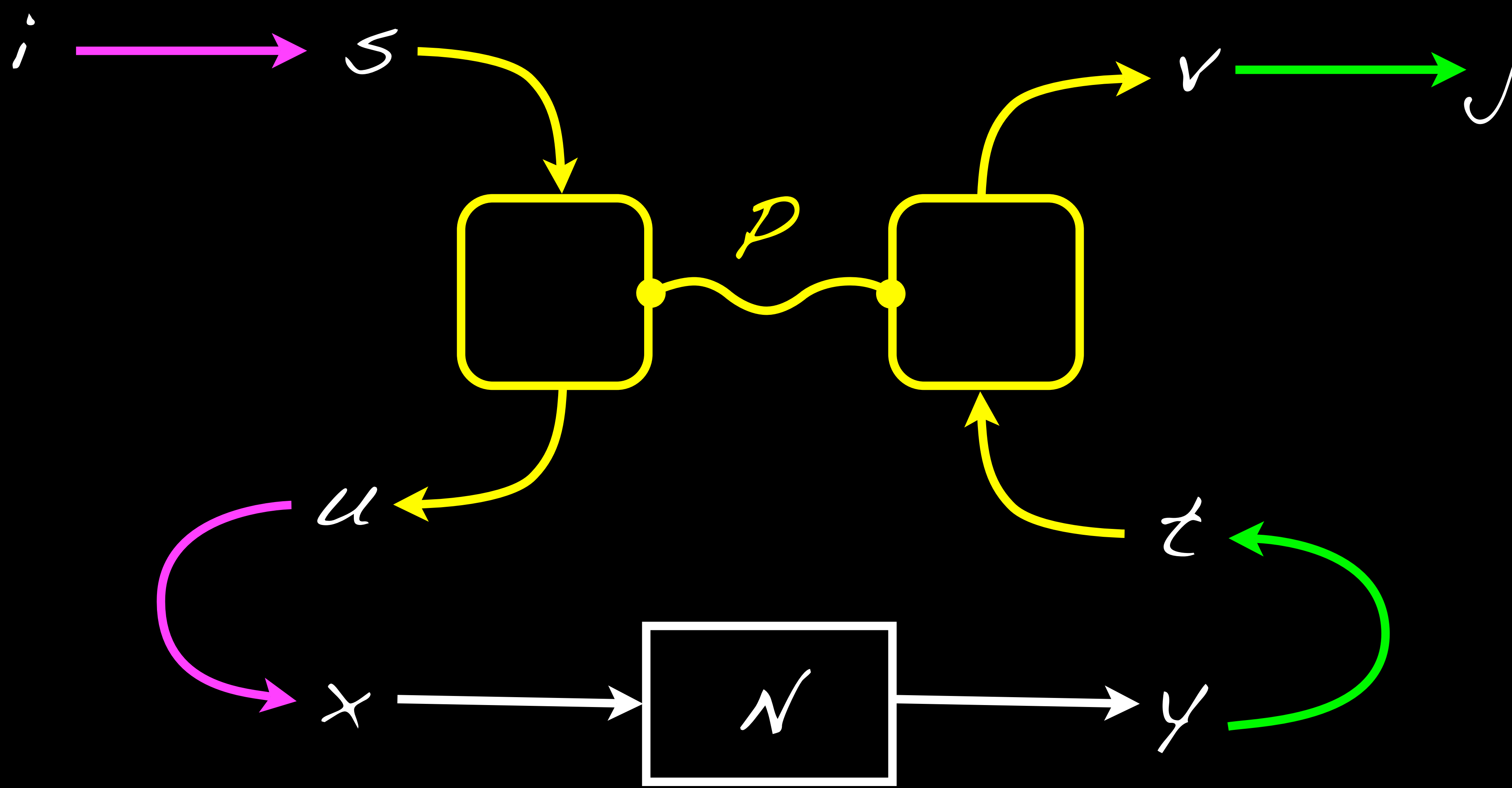


Def: Above mappings form a zero-error code assisted by P if $j=i$ with prob. 1.



Def: Above mappings form a zero-error code assisted by P if $j=i$ with prob. 1.

I.e.: For all $j \neq i$ & edges xy in Γ, $P'(xj|iy)=0$



Def: Above mappings form a zero-error code assisted by P if $j=i$ with prob. 1.

I.e.: For all $j \neq i$ & edges xy in Γ, $P'(xj|iy)=0$

Maximum #msg. with $P \in NS =: \overline{\alpha}(\Gamma)$

$$\overline{\alpha}(\Gamma) = \max m \text{ s.t. } P(xj|iy) \in NS,\ i,j=1\ldots m,$$
$$\forall i\neq j\ \forall xy \in \Gamma \quad P(xj|iy)=0.$$

$$\overline{\alpha}(\Gamma) = \max m \text{ s.t. } P(xj|iy) \in NS,\ ij=1\ldots m,$$

$$\forall i \neq j\ \forall xy \in \Gamma \quad P(xj|iy)=0.$$

Clear: Can test given m efficiently by linear programming. Less obvious:

$\overline{\alpha}(\Gamma)$ = max m s.t. $P(xj|iy) \in NS$, $ij=1\ldots m$,

$\forall i\neq j \forall xy \in \Gamma \quad P(xj|iy)=0$.

Clear: Can test given m efficiently by linear programming. Less obvious:

Thm. $\overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$, with α^* the fractional packing number of Γ:

$$\alpha^*(\Gamma) = \max \sum_x w_x \text{ s.t. } w_x \geq 0 \text{ \& for all } y, \quad \sum_x \Gamma(y|x) w_x \leq 1.$$

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011]

Thm. $\overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$, with α^* the fractional packing number of Γ:

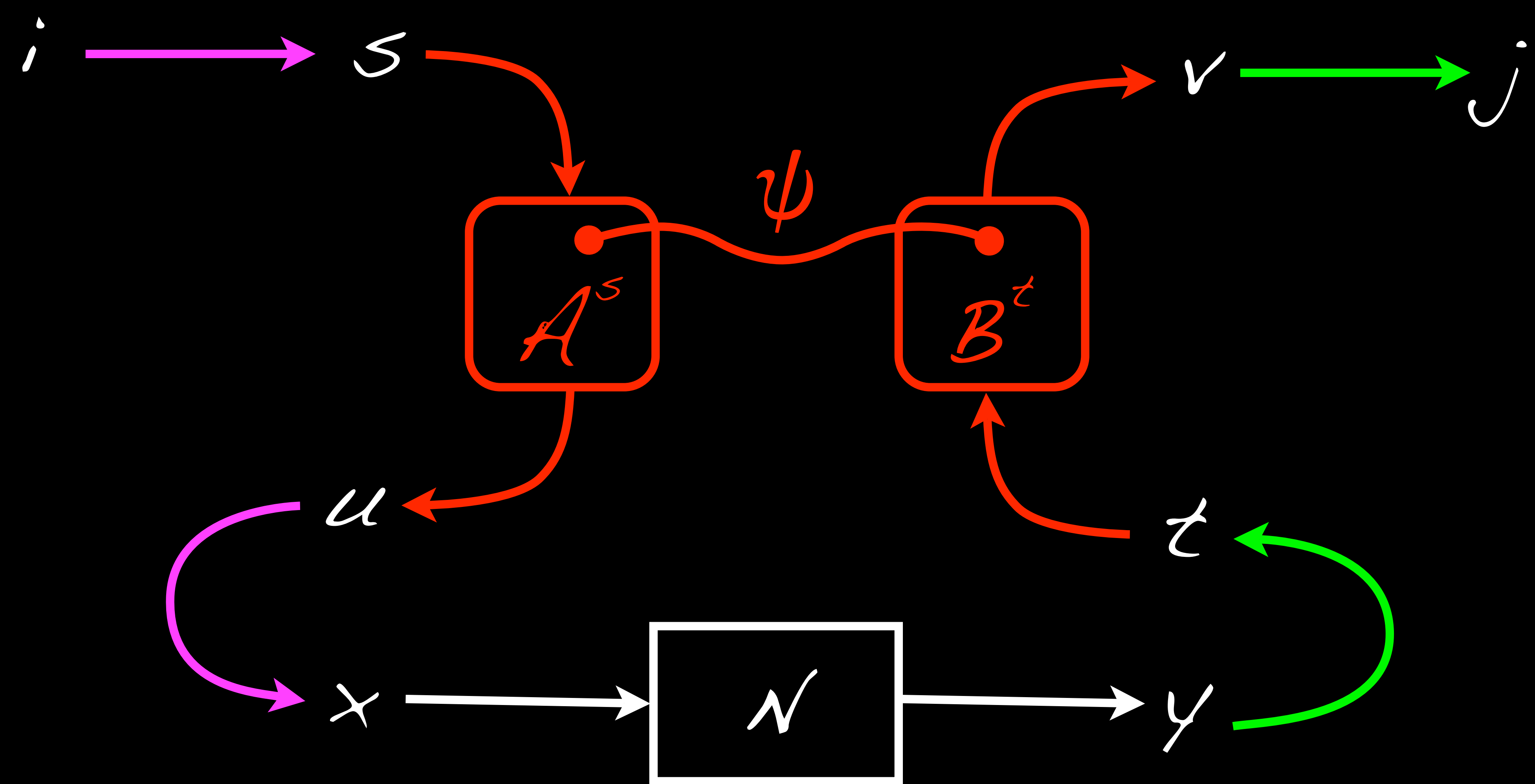
$$\alpha^*(\Gamma) = \max \sum_x w_x \text{ s.t. } w_x \geq 0 \text{ \& for all } y, \quad \sum_x \Gamma(y|x) w_x \leq 1.$$

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011]

Fractional packing number already studied by Shannon [IRE-IT 2(3):8-19, 1956]; it has many nice properties, e.g. multiplicativity:

$$\alpha^*(\Gamma_1 \otimes \Gamma_2) = \alpha^*(\Gamma_1)\, \alpha^*(\Gamma_2)$$

Analogously, if the maximum is over quantum correlations:



Maximum #msg. with $P \in Q =: \widetilde{\alpha}(\Gamma)$

3. Operator characterization

β and ϑ

Unlike the no-signalling variant $\overline{\alpha}$, the entanglement-assisted independence number $\tilde{\alpha}$ depends only on G...

Unlike the no-signalling variant $\overline{\alpha}$, the entanglement-assisted independence number $\tilde{\alpha}$ depends only on G:

Prop. $\tilde{\alpha}(\Gamma) = \tilde{\alpha}(G)$ = max m s.t. exist operators S, S_x^i $(i=1\ldots m)$ with

0) $\operatorname{Tr} SS^\dagger = 1$

1) $\sum_x S_x^i = S \quad \forall i$

2) $S_x^i S_y^{i\dagger} = 0 \quad \forall i \forall x \neq y$

3) $S_x^{i\dagger} S_y^j = 0 \quad \forall i \neq j \forall x \sim y$

[T.S. Cubitt/S. Severini/AW, manuscript, 2011]

Could take this as definition:

$\tilde{\alpha}(G) = \max m$ s.t. exist S, S_x^i $(i=1\ldots m)$ with

0) $\mathrm{Tr}\, SS^\dagger = 1$

1) $\sum_x S_x^i = S \quad \forall i$

2) $S_x^i S_y^{i\dagger} = 0 \quad \forall i \forall x \neq y$

3) $S_x^{i\dagger} S_y^j = 0 \quad \forall i \neq j \forall x \sim y$

Could take this as definition:

$\tilde{\alpha}(G)$ = max m s.t. exist S, S_x^i ($i=1\ldots m$) with

0) $\mathrm{Tr}\, S S^\dagger = 1$

1) $\sum_x S_x^i = S \quad \forall i$

2) $S_x^i S_y^{i\dagger} = 0 \quad \forall i \forall x \neq y$

3) $S_x^{i\dagger} S_y^j = 0 \quad \forall i \neq j \forall x \sim y$

Note: Replacing operators ("q-numbers") with scalars ("c-numbers") recovers α!

And $\{x : S_x^i \neq 0$ for some $i\}$ is indep. set :-)

$\tilde{\alpha}(G)$ = max m s.t. exist S, S_x^i $(i=1\ldots m)$ with

0) $\mathrm{Tr}\, SS^\dagger = 1$

1) $\sum_x S_x^i = S \quad \forall i$

2) $S_x^i S_y^{i\dagger} = 0 \quad \forall i \forall x \neq y$

3) $S_x^{i\dagger} S_y^j = 0 \quad \forall i \neq j \forall x \sim y$

$\tilde{\alpha}(G)$ = max m s.t. exist S, S_x^i $(i=1 \ldots m)$ with

0) $\mathrm{Tr}\, SS^\dagger = 1$

1) $\sum_x S_x^i = S \quad \forall i$

2) $S_x^i S_y^{i\dagger} = 0 \quad \forall i \forall x \neq y$

3) $S_x^{i\dagger} S_y^j = 0 \quad \forall i \neq j \forall x \sim y$

$$\alpha(G) \leq \tilde{\alpha}(G) \leq \overline{\alpha}(\Gamma)$$

$\tilde{\alpha}(G)$ = max m s.t. exist S, S_x^i $(i=1\ldots m)$ with

$$0)\ \mathrm{Tr}\, SS^\dagger = 1$$

$$1)\ \sum_x S_x^i = S \quad \forall i$$

$$2)\ S_x^i S_y^{i\dagger} = 0 \quad \forall i \forall x \neq y$$

$$3)\ S_x^{i\dagger} S_y^j = 0 \quad \forall i \neq j \forall x \sim y$$

$$\alpha(G) \leq \tilde{\alpha}(G) \leq \overline{\alpha}(\Gamma)$$

Note: α and $\overline{\alpha}$ computable (NP-hard and polynomial, respectively);

$\tilde{\alpha}$ not known to be computable...

$\tilde{\alpha}(G)$ = max m s.t. exist S, S_x^i $(i=1\ldots m)$ with

0) $Tr\, SS^\dagger = 1$

1) $\sum_x S_x^i = S \quad \forall i$

2) $S_x^i S_y^{i\dagger} = 0 \quad \forall i \forall x \neq y$

3) $S_x^{i\dagger} S_y^j = 0 \quad \forall i \neq j \forall x \sim y$

$$\alpha(G) \leq \tilde{\alpha}(G) \leq \overline{\alpha}(\Gamma)$$

Note: α and $\overline{\alpha}$ computable (NP-hard and polynomial, respectively);

$\tilde{\alpha}$ $\Rightarrow$ Look for relaxations!

$\tilde{\alpha}(G) = \max m$ s.t. exist S, S_x^i $(i=1\ldots m)$ with

$$0)\quad \mathrm{Tr}\, SS^\dagger = 1$$

$$1)\quad \sum_x S_x^i = S \quad \forall i$$

$$2)\quad S_x^i S_y^{i\dagger} = 0 \quad \forall i \forall x \neq y$$

$$3)\quad S_x^{i\dagger} S_y^j = 0 \quad \forall i \neq j \forall x \sim y$$

$\leq \beta(G) := \max n$ s.t. vectors v, v_x^i $(i=1\ldots n)$

$$0)\quad \langle v, v\rangle = 1$$

$$1)\quad \sum_x v_x^i = v \quad \forall i$$

$$2)\quad \langle v_x^i, v_y^i \rangle = 0 \quad \forall i \forall x \neq y$$

$$3)\quad \langle v_x^i, v_y^j \rangle = 0 \quad \forall i \neq j \forall x \sim y$$

$\tilde{\alpha}(G) \leq \beta(G)$ [S. Beigi, PRA 82:010303, 2010]

$:=$ max n s.t. vectors v, v_x^i $(i=1\ldots n)$

0) $\langle v,v\rangle = 1$

1) $\sum_x v_x^i = v \quad \forall i$

2) $\langle v_x^i, v_y^i\rangle = 0 \quad \forall i \forall x \neq y$

3) $\langle v_x^i, v_y^j\rangle = 0 \quad \forall i \neq j \forall x \sim y$

$\tilde{\alpha}(G) \leq \beta(G)$ [S. Beigi, PRA 82:010303, 2010]

$:= \max n$ s.t. vectors v, v_x^i $(i=1\ldots n)$

0) $\langle v, v \rangle = 1$

1) $\sum_x v_x^i = v \quad \forall i$

2) $\langle v_x^i, v_y^i \rangle = 0 \quad \forall i \forall x \neq y$

3) $\langle v_x^i, v_y^j \rangle = 0 \quad \forall i \neq j \forall x \sim y$

Refers to elements of a Gram matrix, subject to positivity.

$\tilde{\alpha}(G) \leq \beta(G)$ [S. Beigi, PRA 82:010303, 2010]

$:= \max n$ s.t. vectors v, v_x^i $(i=1\ldots n)$

0) $\langle v,v\rangle = 1$

1) $\sum_x v_x^i = v \quad \forall i$

2) $\langle v_x^i, v_y^i\rangle = 0 \quad \forall i \forall x\neq y$

3) $\langle v_x^i, v_y^j\rangle = 0 \quad \forall i\neq j \forall x\sim y$

Refers to elements of a Gram matrix, subject to positivity.

Hence, fixed n can be checked by SDP ✓

$\tilde{\alpha}(G) \leq \beta(G)$ [S. Beigi, PRA 82:010303, 2010]

:= max n s.t. vectors v, v_x^i $(i=1\ldots n)$

0) $\langle v, v\rangle = 1$

1) $\sum_x v_x^i = v \quad \forall i$

2) $\langle v_x^i, v_y^i\rangle = 0 \quad \forall i \forall x \neq y$

3) $\langle v_x^i, v_y^j\rangle = 0 \quad \forall i \neq j \forall x \sim y$

Thm. $\beta(G) = \lfloor \vartheta(G) \rfloor$, with Lovász'

$\vartheta(G) = \max \operatorname{Tr} BJ$ s.t. $B \geq 0$, $\operatorname{Tr} B = 1$,

$B_{xy} = 0 \ \forall xy \in G$.

[L. Lovász, IEEE-IT 25(1):1-7, 1979;

T.S. Cubitt/S. Severini/AW, manuscript, 2011.]

$$\alpha(G) \leq \widetilde{\alpha}(G) \leq \beta(G) \leq \overline{\alpha}(\Gamma)$$

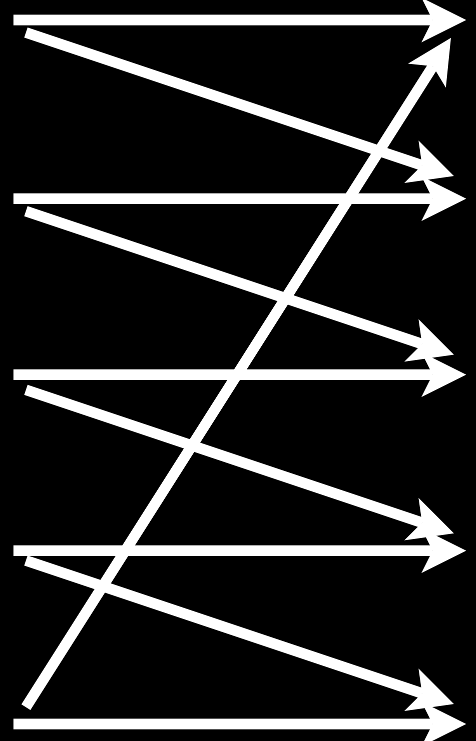
$$\alpha(G) \leq \tilde{\alpha}(G) \leq \beta(G) \leq \overline{\alpha}(\Gamma)$$

$$\beta(G) = \lfloor \vartheta(G) \rfloor \qquad \overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$$

$$\alpha(G) \le \tilde{\alpha}(G) \le \beta(G) \le \overline{\alpha}(\Gamma)$$

$$\tilde{\alpha}(G) = \lfloor \vartheta(G) \rfloor \qquad \overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$$

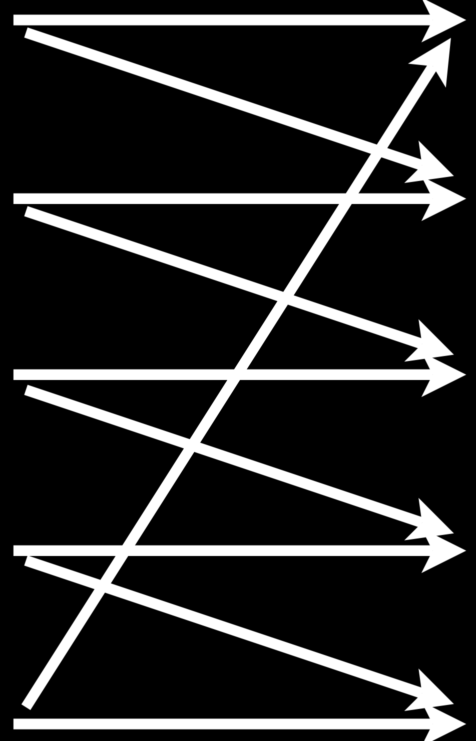
Ex. Typewriter channel/pentagon



$$\alpha(G) \leq \widetilde{\alpha}(G) \leq \beta(G) \leq \overline{\alpha}(\Gamma)$$

$$\beta(G) = \lfloor \vartheta(G) \rfloor \qquad \overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$$

Ex. Typewriter channel/pentagon



All independence numbers 2,
but $\vartheta(G) = \sqrt{5}$ and $\alpha^*(\Gamma) = 5/2$.

$$\alpha(G) \le \widetilde{\alpha}(G) \le \beta(G) \le \overline{\alpha}(\Gamma)$$

$$\lfloor \vartheta(G) \rfloor \qquad \lfloor \alpha^*(\Gamma) \rfloor$$

(with "=" marks under $\beta(G)$ and $\overline{\alpha}(\Gamma)$, linking them to $\lfloor \vartheta(G) \rfloor$ and $\lfloor \alpha^*(\Gamma) \rfloor$ respectively)

SDP → $\lfloor \vartheta(G) \rfloor$

LP → $\lfloor \alpha^*(\Gamma) \rfloor$

$$\alpha(G) \le \tilde{\alpha}(G) \le \beta(G) \le \overline{\alpha}(\Gamma)$$

$$\beta(G) = \lfloor \vartheta(G) \rfloor \qquad \overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$$

SDP

LP

Question: "Operational" characterization of β, e.g. via a correlation class?

$$\alpha(G) \le \tilde{\alpha}(G) \le \beta(G) \le \overline{\alpha}(\Gamma)$$

$$\beta(G) = \lfloor \vartheta(G) \rfloor \qquad \overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$$

Since ϑ is multiplicative under strong graph product, $\vartheta(G \times H) = \vartheta(G)\vartheta(H)$, get:

$$C_{0E}(G) = \lim \frac{1}{n} \log \tilde{\alpha}(G^{\times n}) \le \log \vartheta(G)$$

$$\alpha(G) \leq \tilde{\alpha}(G) \leq \beta(G) \leq \overline{\alpha}(\Gamma)$$

$$\beta(G) = \lfloor \vartheta(G) \rfloor \qquad \overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$$

Since ϑ is multiplicative under strong graph product, $\vartheta(G \times H) = \vartheta(G)\vartheta(H)$, get:

$$C_0(G) \leq C_{0\varepsilon}(G) = \lim \frac{1}{n} \log \tilde{\alpha}(G^{\times n}) \leq \log \vartheta(G)$$

$$\alpha(G) \le \tilde{\alpha}(G) \le \beta(G) \le \overline{\alpha}(\Gamma)$$

$$\beta(G) = \lfloor \vartheta(G) \rfloor \qquad \overline{\alpha}(\Gamma) = \lfloor \alpha^*(\Gamma) \rfloor$$

Since ϑ is multiplicative under strong graph product, $\vartheta(G \times H) = \vartheta(G)\vartheta(H)$, get:

$$C_0(G) \le C_{0E}(G) = \lim \frac{1}{n} \log \tilde{\alpha}(G^{\times n}) \le \log \vartheta(G)$$

$$C_{0NS}(\Gamma) = \lim \frac{1}{n} \log \overline{\alpha}(\Gamma^{\otimes n}) = \log \alpha^*(\Gamma)$$

4. Entanglement can make a difference

$$\alpha(G) \leq \tilde{\alpha}(G) \leq \vartheta(G)$$

$$C_0(G) \leq C_{0E}(G) \leq \log \vartheta(G)$$

4. Entanglement can make a difference

$$\alpha(G) \leq \tilde{\alpha}(G) \leq \vartheta(G)$$

$$C_0(G) \leq C_{0E}(G) \leq \log \vartheta(G)$$

Both can be strict

[T.S. Cubitt/D. Leung/W. Matthews/AW, PRL 104:230503, 2010; D. Leung/L. Mancinska/W. Matthews/+2, CMP 311:97-111, 2012, using W. Haemers, IEEE-IT 25(2):231-232, 1979.]

4. Entanglement can make a difference

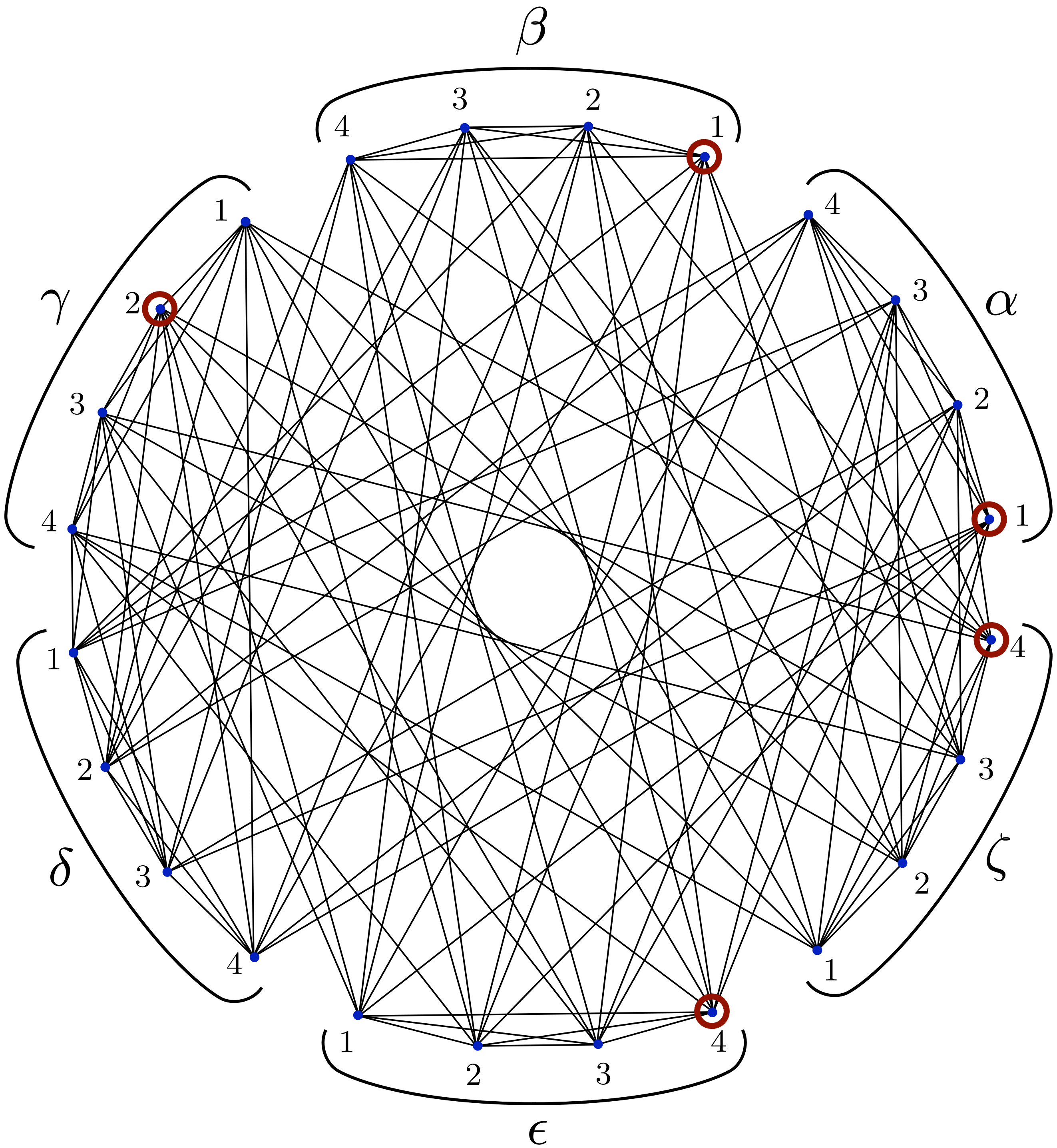
$$\alpha(G) \le \tilde{\alpha}(G) \le \vartheta(G)$$

$$C_0(G) \le C_{0E}(G) \le \log \vartheta(G)$$

Both can be strict

$= \lfloor \vartheta(G) \rfloor$ in all these examples

[T.S. Cubitt/D. Leung/W. Matthews/AW, PRL 104:230503, 2010;
D. Leung/L. Mancinska/W. Matthews/+2, CMP 311:97–111, 2012,
using W. Haemers, IEEE-IT 25(2):231–232, 1979.]



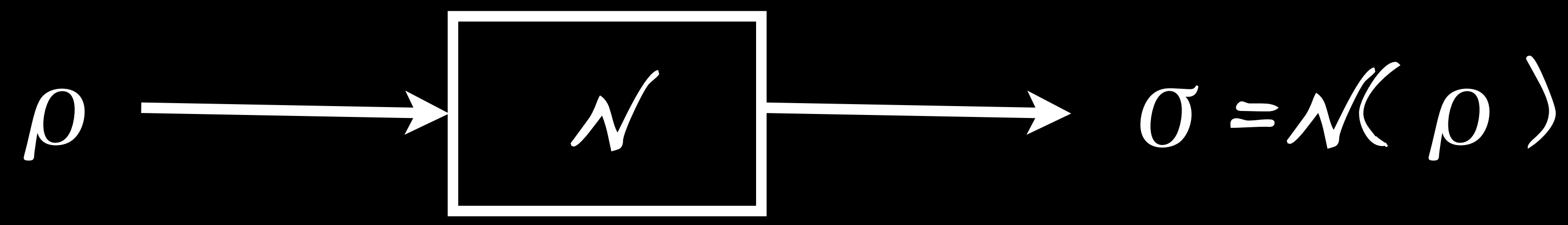
Peres-Mermin graph (orthogonality graph of a specific Kochen-Specker set).

	1	2	3	4
α	$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,1,0)$	$(0,0,0,1)$
β	$(-1,1,1,1)$	$(1,-1,1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$
γ	$(1,1,1,1)$	$(1,1,-1,-1)$	$(1,-1,1,-1)$	$(1,-1,-1,1)$
δ	$(1,1,0,0)$	$(1,-1,0,0)$	$(0,0,1,1)$	$(0,0,1,-1)$
ϵ	$(0,1,1,0)$	$(0,1,-1,0)$	$(1,0,0,1)$	$(1,0,0,-1)$
ζ	$(0,1,0,1)$	$(0,1,0,-1)$	$(1,0,1,0)$	$(1,0,-1,0)$

[T.S. Cubitt/D. Leung/W. Matthews/AW, PRL 104:230503, 2010]

5. Going all the quantum way...

Should really consider quantum channels $\mathcal{N}: B(A) \longrightarrow B(B)$:



[A.S. Holevo, Statistical Structure of Quantum Theory, Springer LNP 67, 2001].

For quantum channel (cptp map) $N: B(A) \longrightarrow B(B)$, with Kraus op's E_i:

Define $K = \mathrm{span}\{E_i\} \subset B(A \to B)$ and $S = K^\dagger K = \mathrm{span}\{E_i^\dagger E_j\} \subset B(A)$ as natural analogues of the transition/confusability graph.

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013.]

For quantum channel (cptp map) $N: B(A) \longrightarrow B(B)$, with Kraus op's E_i:

Define $K = \mathrm{span}\{E_i\} \subset B(A \to B)$ and $S = K^\dagger K = \mathrm{span}\{E_i^\dagger E_j\} \subset B(A)$ as natural analogues of the transition/confusability graph.

(For classical channel: S consists of those $X \times X$-matrices with xx'-entry non-zero only if $x \sim x'$ in G; similar for K.)

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013.]

Define $K = \text{span}\{E_i\} \subset B(A \to B)$ and

$S = K^\dagger K = \text{span}\{E_i^\dagger E_j\} \subset B(A)$ as natural analogues of the transition/confusability graph.

Zero-error transmission assisted by entanglement depends only on S; assisted by no-signalling only on K.

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013;
R. Duan/AW, in preparation.]

➡ Definitions of $\alpha(S)$, $\tilde{\alpha}(S)$ and $\overline{\alpha}(K)$, again via maximal number of messages. [Reducing to previous notions in the classical case.]

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013;
R. Duan/AW, in preparation.]

➡ Definitions of $\alpha(S)$, $\tilde{\alpha}(S)$ and $\overline{\alpha}(K)$, again via maximal number of messages. [Reducing to previous notions in the classical case.]

Thm. $\overline{\alpha}(K) = \lfloor \alpha^*(K) \rfloor$; the number on the rhs is a semidefinite program, generalizing the fractional packing no.

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013;
R. Duan/AW, in preparation.]

➡ Definitions of $\alpha(S)$, $\tilde{\alpha}(S)$ and $\overline{\alpha}(K)$, again via maximal number of messages. [Reducing to previous notions in the classical case.]

Thm. $\overline{\alpha}(K) = \lfloor \alpha^*(K) \rfloor$; the number on the rhs is a semidefinite program, generalizing the fractional packing no.

Thm. $\alpha(S)$ and $\tilde{\alpha}(S)$ are upper bounded by $\vartheta(S)$ and $\tilde{\vartheta}(S)$, generalizing $\vartheta(G)$.

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013;
R. Duan/AW, in preparation.]

$$\vartheta(S) = \max \|\mathbb{1} + T\|$$

$$\text{s.t. } T \perp S,$$

$$\mathbb{1} + T \geq 0.$$

$$\vartheta(S) = \max \|\mathbb{1}+T\|$$

$$\text{s.t. } T \perp S,$$
$$\mathbb{1}+T \geq 0.$$

$$\tilde{\vartheta}(S) = \vartheta(S \otimes B(\mathcal{H}_A))$$

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164–1174, 2013]

$$\vartheta(S) = \max \|\mathbb{1}+T\|$$

$$\text{s.t. } T \perp S, \quad \mathbb{1}+T \geq 0.$$

$$\tilde{\vartheta}(S) = \vartheta(S \otimes B(\mathcal{H}_A))$$

Not immediate, but an SDP, and multiplicative;

[R. Duan/S.Severini/AW, IEEE-IT 59(2):1164-1174, 2013]

$$\vartheta(S) = \max \|\mathbb{1}+T\|$$

$$\text{s.t. } T \perp S,$$

$$\mathbb{1}+T \geq 0.$$

$$\tilde{\vartheta}(S) = \vartheta(S \otimes B(\mathcal{H}_A))$$

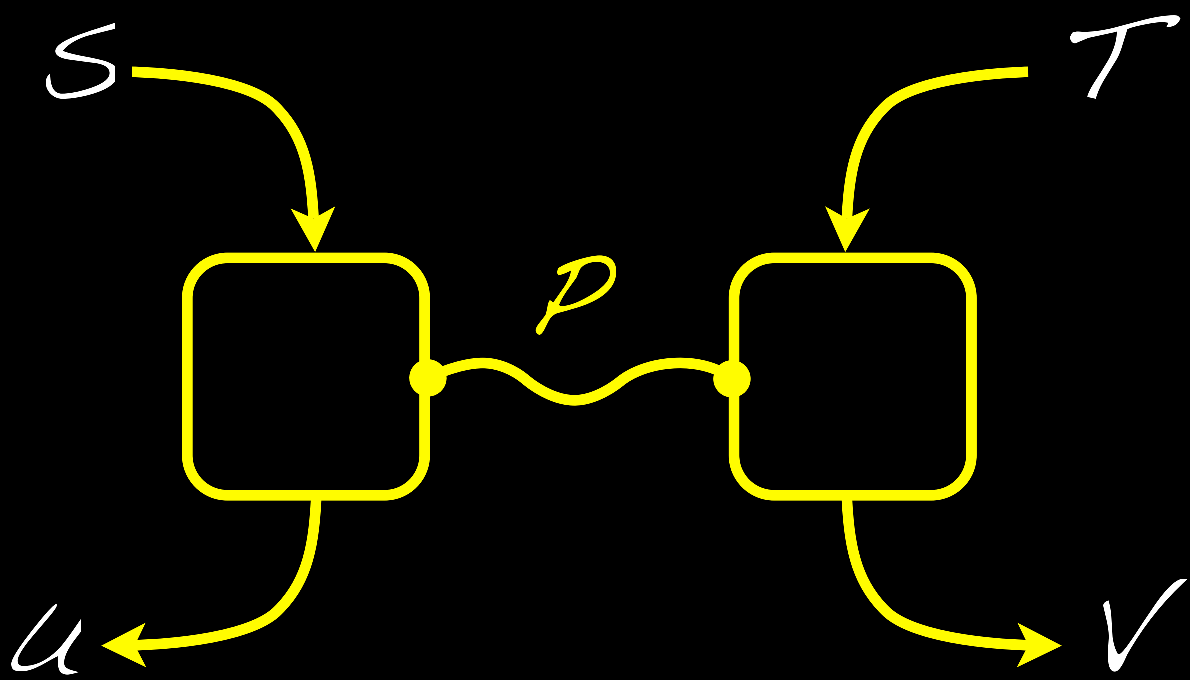
Not immediate, but an SDP, and multiplicative;

$C_{0E}(S) \leq \log \tilde{\vartheta}(S)$.

[R. Duan/S. Severini/AW, IEEE-IT 59(2):1164-1174, 2013]

Quantum no-signalling:

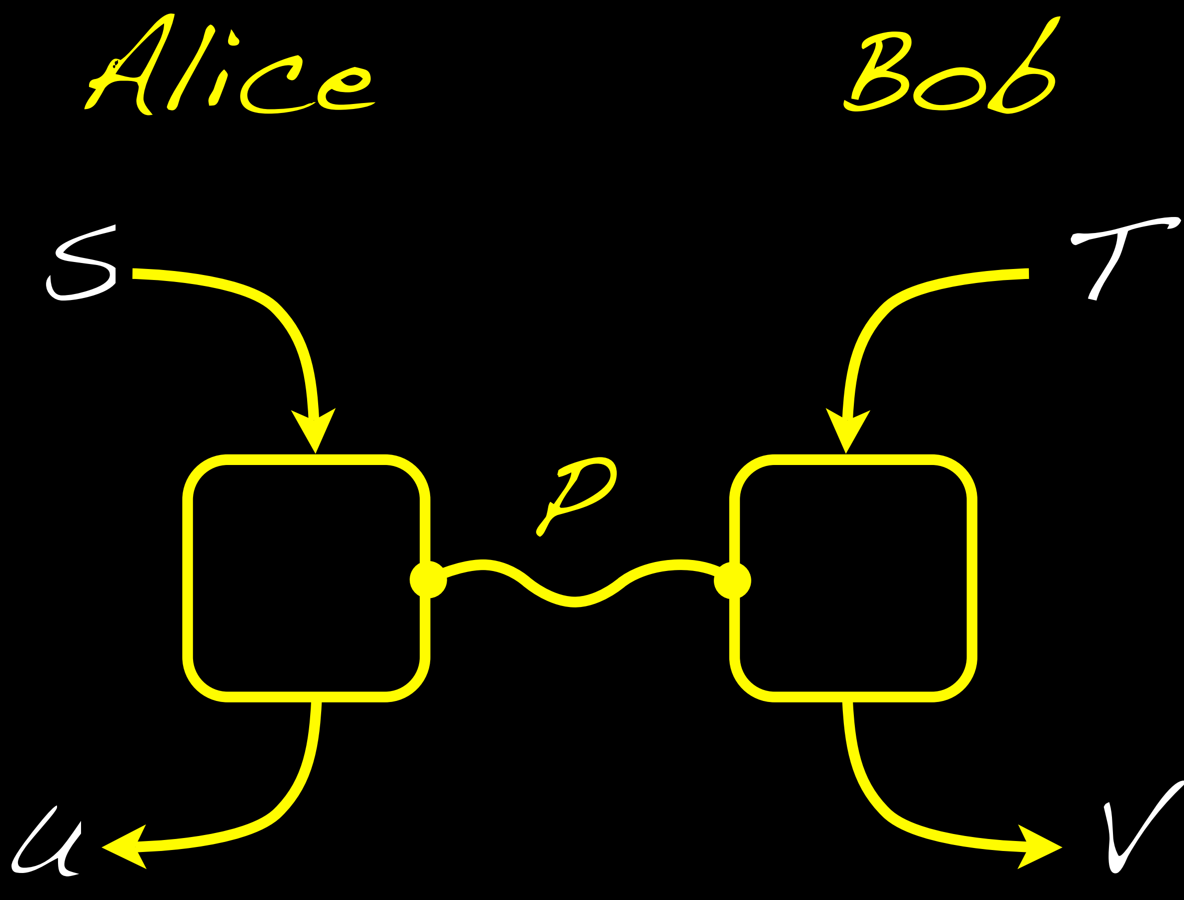
Alice Bob



$P: S \otimes T \to U \otimes V$ cptp

[D. Beckman et al., PRA 64:052309, 2001; T. Eggeling et al., Europhy. Lett. 57(6):782–788, 2002; M. Piani et al., PRA 74:012305, 2006]

Quantum no-signalling:



$P: S \otimes T \to U \otimes V$ cptp

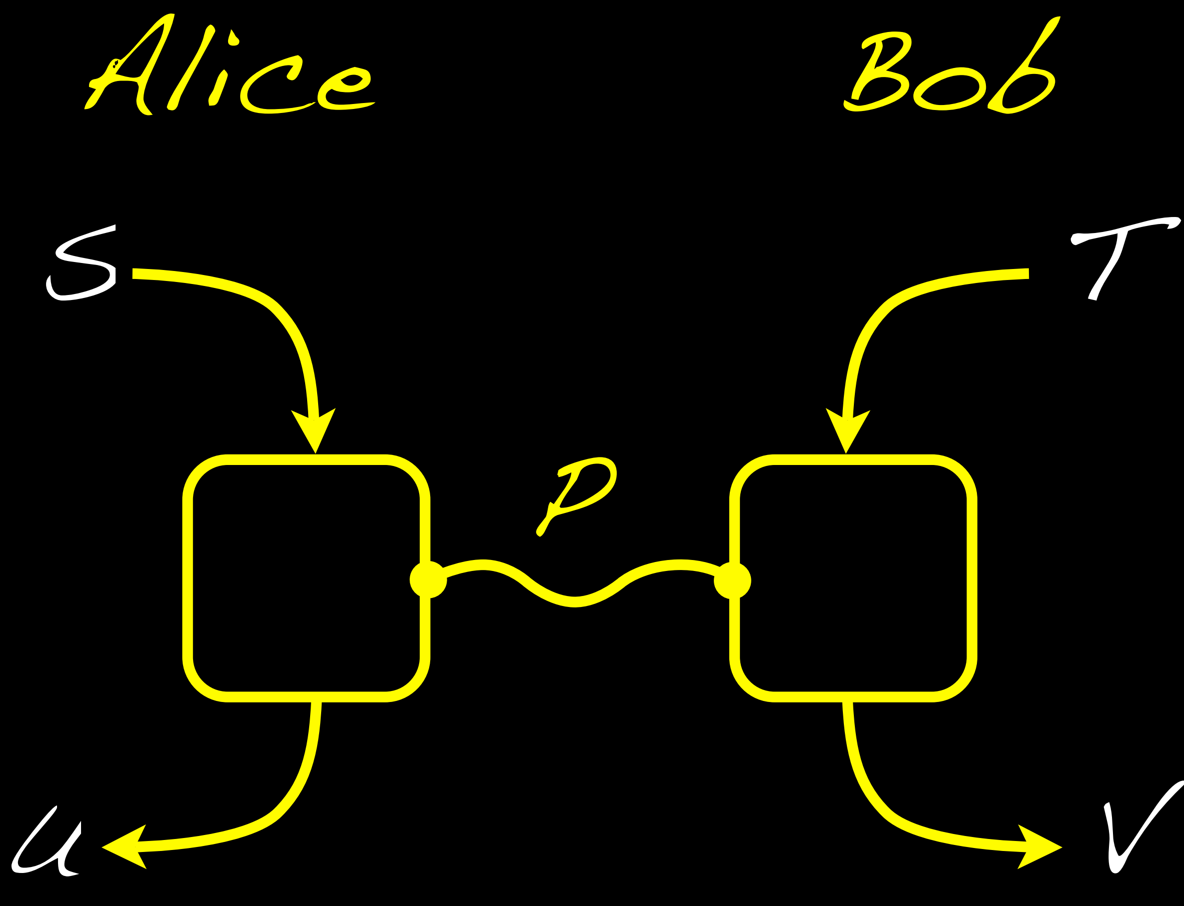
No-signalling means:

$$\mathrm{Tr}_U\, P(\sigma \otimes \tau) = A(\tau),$$

$$\mathrm{Tr}_V\, P(\sigma \otimes \tau) = B(\sigma).$$

[D. Beckman et al., PRA 64:052309, 2001; T. Eggeling et al., Europhy. Lett. 57(6):782-788, 2002; M. Piani et al., PRA 74:012305, 2006]

Quantum no-signalling:



$P: S \otimes T \to U \otimes V$ cptp

No-signalling means:

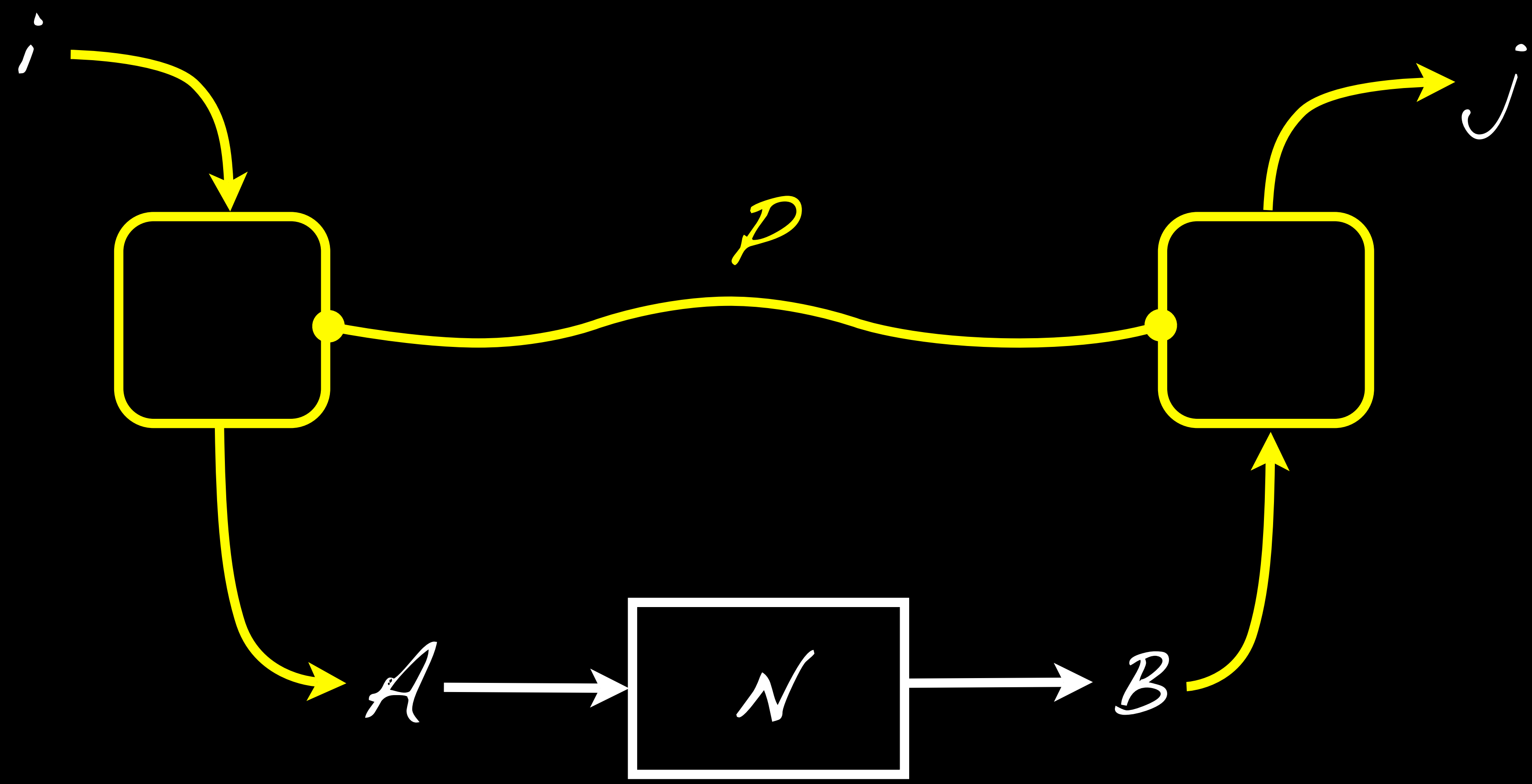
$$\mathrm{Tr}_U\, P(\sigma \otimes \tau) = A(\tau),$$

$$\mathrm{Tr}_V\, P(\sigma \otimes \tau) = B(\sigma).$$

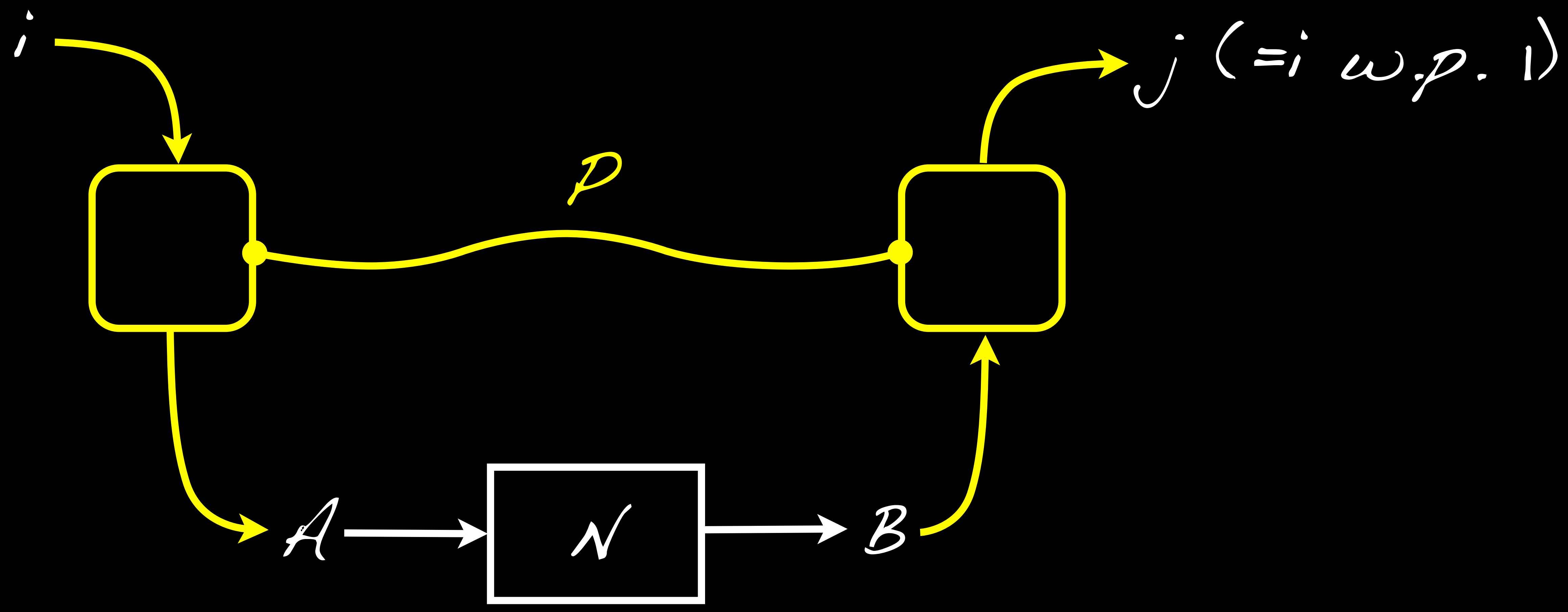
Equivalent: P affine combination of $A_i \otimes B_i$

[D. Beckman et al., PRA 64:052309, 2001; T. Eggeling et al., Europhy. Lett. 57(6):782–788, 2002; M. Piani et al., PRA 74:012305, 2006]

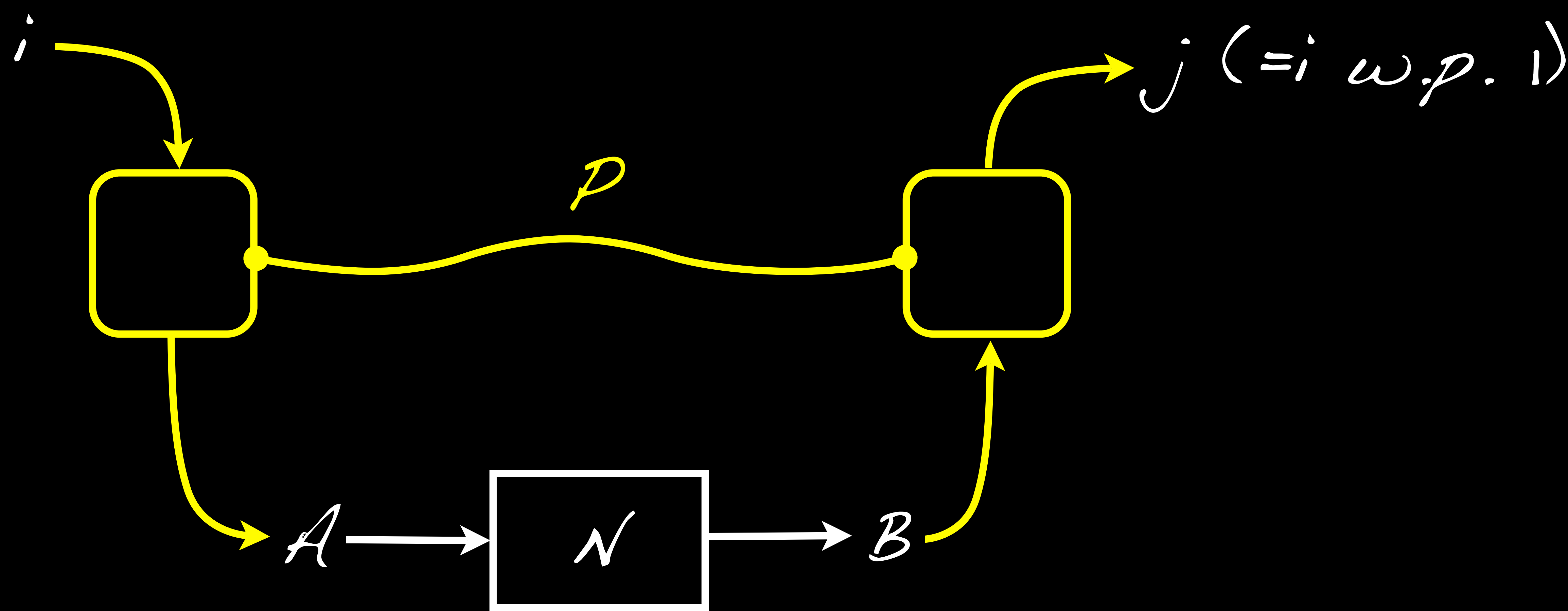
No-signalling assisted communication:



No-signalling assisted communication:



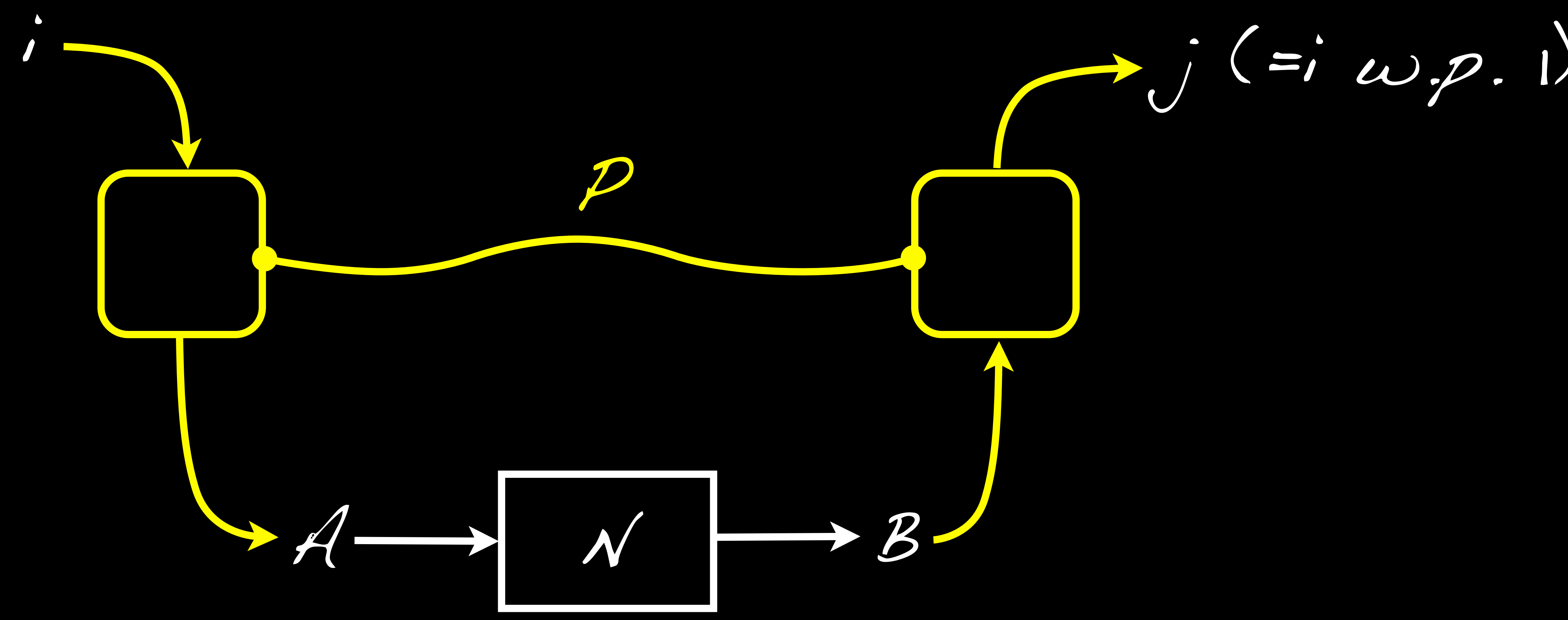
No-signalling assisted communication:



$$\alpha^*(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes \mathbb{1},$$
$$\operatorname{Tr}_A U = \mathbb{1}^B,$$
$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

[R. Duan/AW, in preparation.]

No-signalling assisted communication:

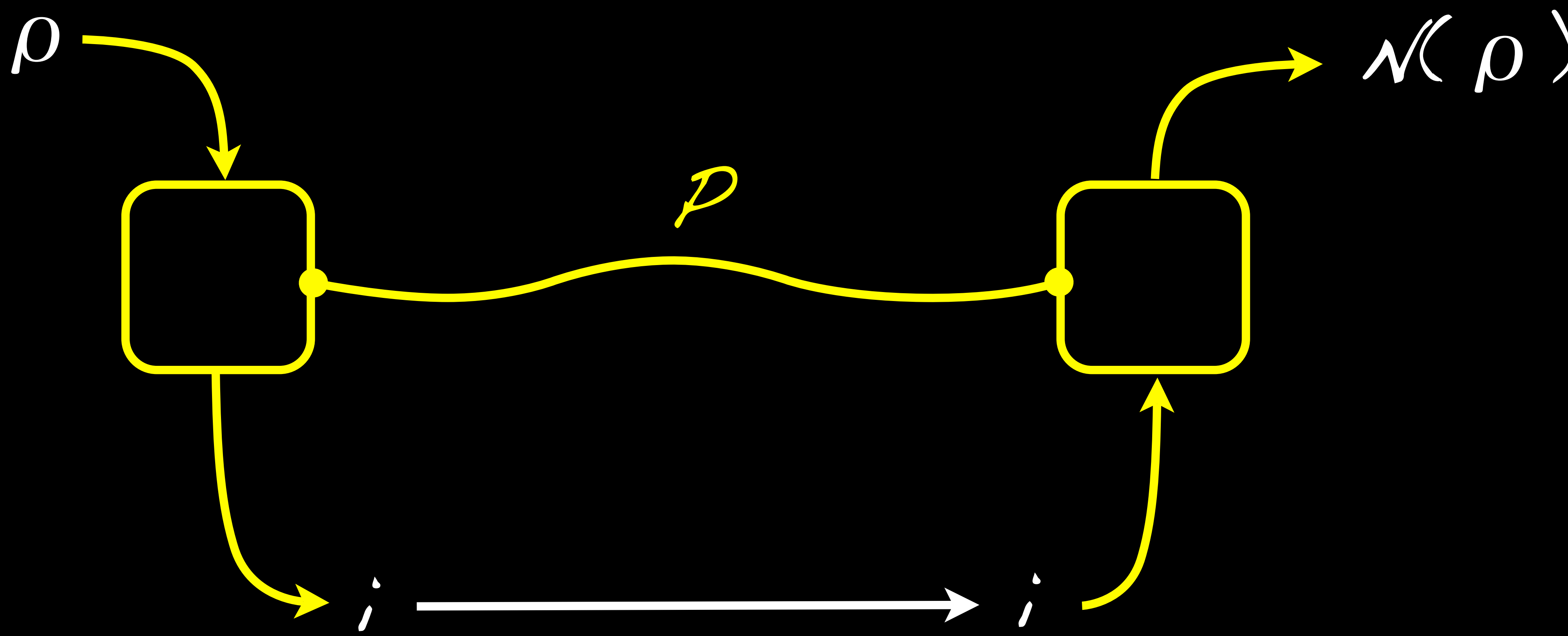


$$\alpha^*(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes \mathbb{1},$$
$$\operatorname{Tr}_A U = \mathbb{1}^B,$$
$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

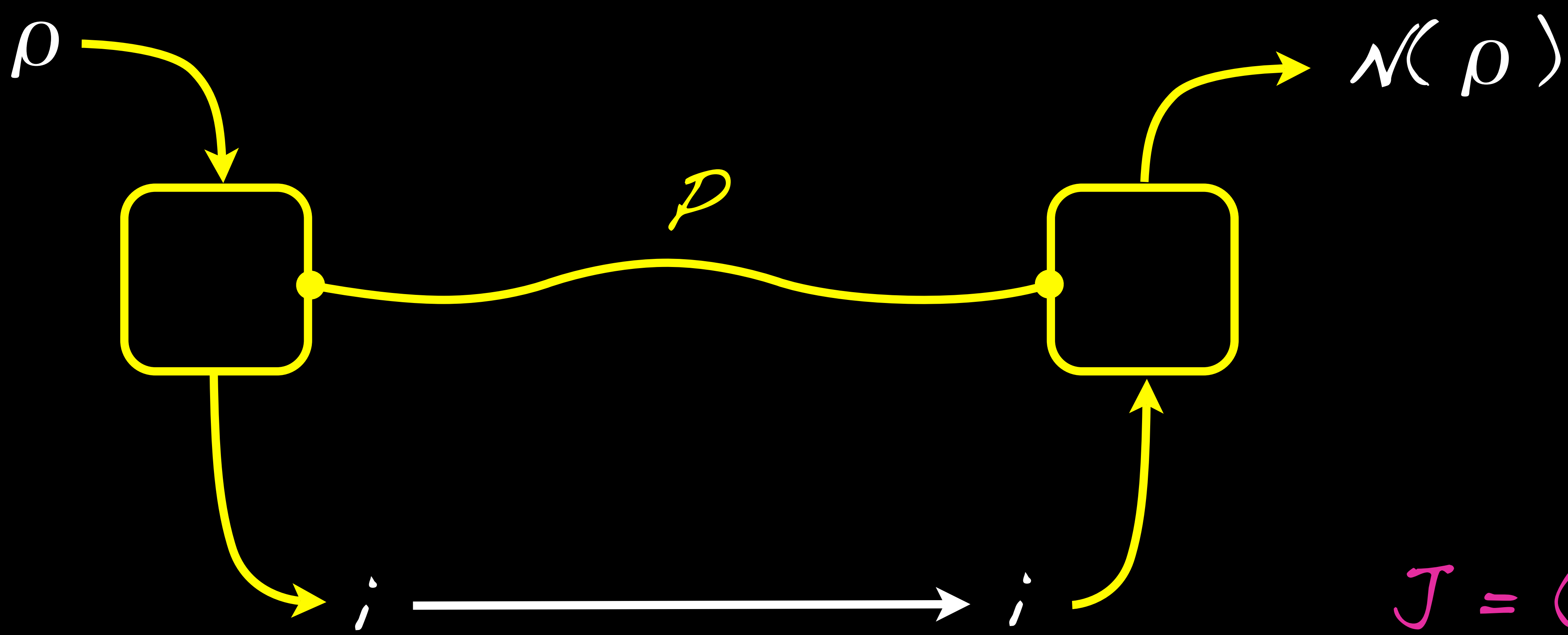
Π: support projection of Choi matrix J of N.

[R. Duan/AW, in preparation.]

No-signalling assisted simulation:

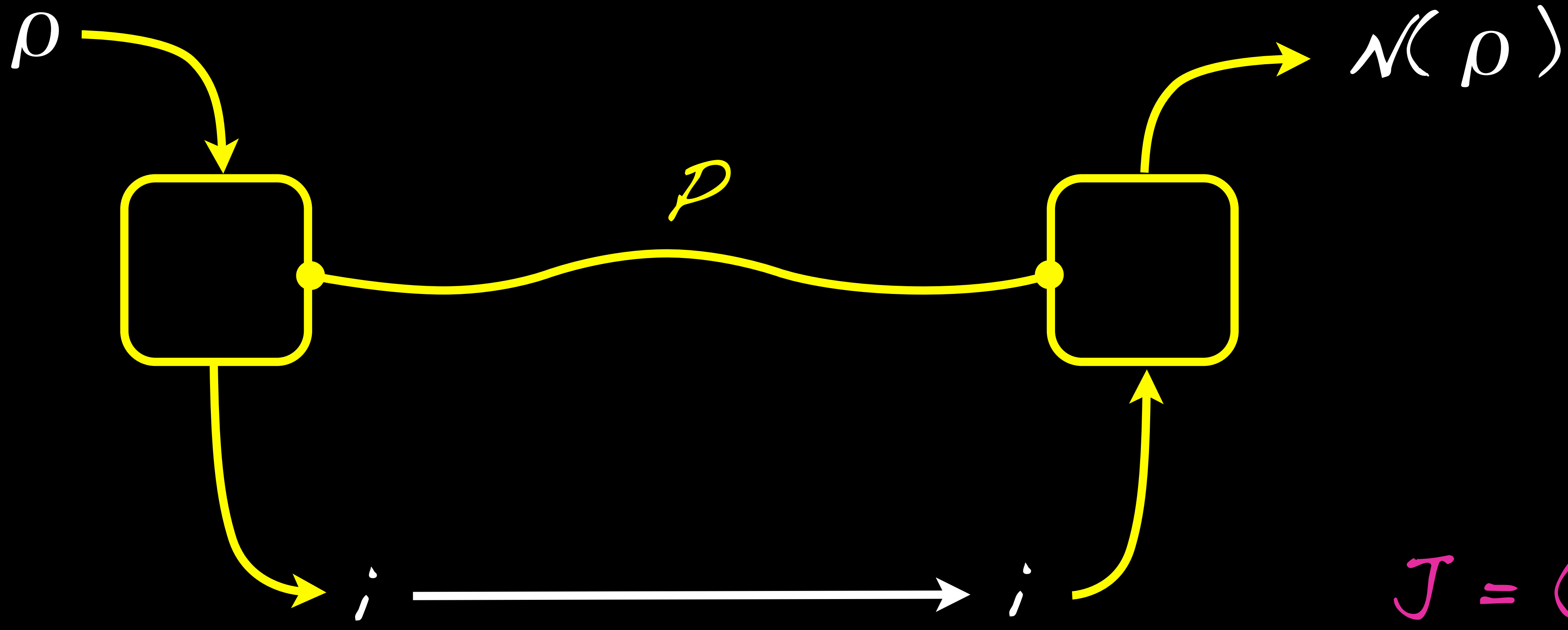


No-signalling assisted simulation:



$J = (id \otimes \mathcal{N})\Phi$

No-signalling assisted simulation:



$J = (id \otimes \mathcal{N})\Phi$

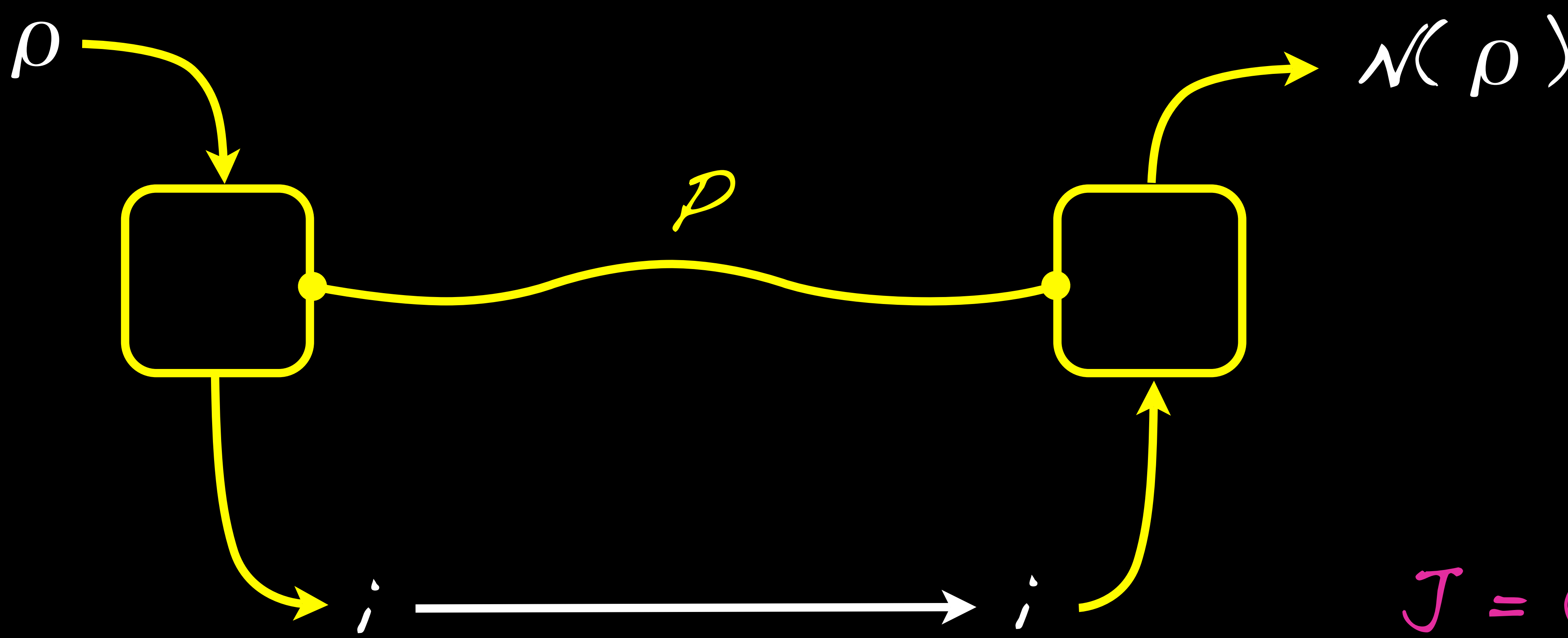
Number of messages: integer part of

$$2^{-H_{min}(A|B)_J} = \max \operatorname{Tr} T$$

$$\text{s.t. } 0 \le J^{AB} \le \mathbb{1} \otimes T$$

[R. Duan/AW, in preparation.]

No-signalling assisted simulation:



$J = (id \otimes \mathcal{N})\Phi$

Number of messages: integer part of

$$2^{-H_{min}(A|B)_J} = \max \operatorname{Tr} T$$

$$\text{s.t. } 0 \leq J^{AB} \leq \mathbb{1} \otimes T$$

Yes, it's Renner's min-entropy!

[R. Duan/AW, in preparation.]

$$\alpha^*(K) = \max \mathrm{Tr}\, S \text{ s.t. } 0 \le U^{AB} \le S \otimes \mathbb{1},$$
$$\mathrm{Tr}_A U = \mathbb{1}^B,$$
$$\Pi(S \otimes \mathbb{1} - U) = 0.$$

...reduces to classical fractional packing number for classical channel.

[R. Duan/AW, in preparation.]

$$\alpha^*(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq U^{AB} \leq S \otimes \mathbb{1}, \quad \operatorname{Tr}_A U = \mathbb{1}^B, \quad \Pi(S \otimes \mathbb{1} - U) = 0.$$

...reduces to classical fractional packing number for classical channel.

As does the minimal simulation cost over all channels with Kraus operators in K:

$$\Sigma(K) = \min \operatorname{Tr} T \text{ s.t. } 0 \leq V^{AB} \leq \mathbb{1} \otimes T, \quad \operatorname{Tr}_B V = \mathbb{1}^A, \quad (\mathbb{1} - \Pi)V = 0.$$

[T.S. Cubitt et al., IEEE-IT 57(8):5509-5523, 2011; R. Duan/AW, in prep.]

However, in general much more complex;
from now on focus on cq-channels...

However, in general much more complex;
from now on focus on cq-channels...

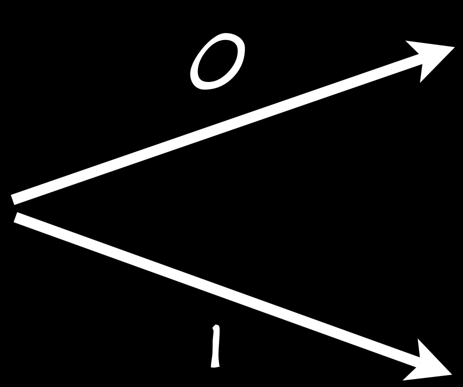
Finitely many output states with support Π_x, $\Pi = \sum_x |x\rangle\langle x| \otimes \Pi_x$.

However, in general much more complex;

from now on focus on cq-channels...

Finitely many output states with support Π_x, $\Pi = \sum_x |x\rangle\langle x| \otimes \Pi_x$.

Example: Two-pure-state cq-channel



$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$

$K = \mathrm{span}\{|\psi_0\rangle\langle 0|, |\psi_1\rangle\langle 1|\}$

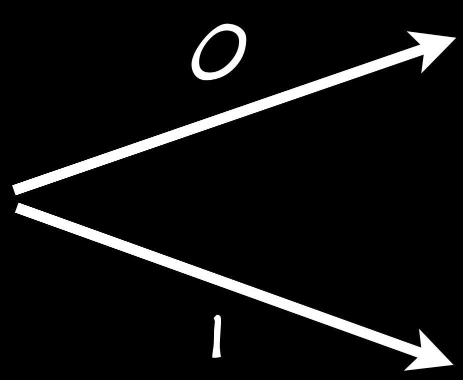
Example: Two-pure-state cq-channel

$$0 \rightarrow |\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$1 \rightarrow |\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$$

$$K = \text{span}\{|\psi_0\rangle\langle 0|,\ |\psi_1\rangle\langle 1|\}$$

Example: Two-pure-state cq-channel



$$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$$

$$K = \mathrm{span}\{|\psi_0\rangle\langle 0|, \; |\psi_1\rangle\langle 1|\}$$

$$\Sigma(K) = 1 + \alpha\beta ;$$

& asymptotic cost equals $\log \Sigma(K)$.

Example: Two-pure-state cq-channel

$0 \to |\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$

$1 \to |\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$

$K = \mathrm{span}\{|\psi_0\rangle\langle 0|, |\psi_1\rangle\langle 1|\}$

$\Sigma(K) = 1 + \alpha\beta$;

& asymptotic cost equals $\log \Sigma(K)$.

$\alpha^*(K) = 1$, but $\alpha^*(K \otimes K) \geq 1/(2\alpha^4)$...

Example: Two-pure-state cq-channel

$0 \to |\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$

$1 \to |\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$

$K = \mathrm{span}\{|\psi_0\rangle\langle 0|, |\psi_1\rangle\langle 1|\}$

$\Sigma(K) = 1 + \alpha\beta$;

& asymptotic cost equals $\log \Sigma(K)$.

$\alpha^*(K) = 1$, but $\alpha^*(K \otimes K) \geq 1/(2\alpha^4)$...

What is $C_{ONS}(K) = \lim \frac{1}{n} \log \alpha^*(K^{\otimes n})$?

$$\alpha^*(K) \leq \mathsf{A}(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq S \leq \mathbb{1},$$

$$\operatorname{Tr}_A \Pi(S \otimes \mathbb{1})\Pi \leq \mathbb{1}.$$

(for cq-channels)

[R. Duan/AW, in preparation.]

$$\alpha^*(K) \leq \mathsf{A}(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq S \leq \mathbb{1},$$

$$\operatorname{Tr}_A \Pi(S \otimes \mathbb{1})\Pi \leq \mathbb{1}.$$

(for cq-channels)

Thm. $C_{ONS}(K) = \lim \frac{1}{n} \log \alpha^*(K^{\otimes n}) = \log \mathsf{A}(K).$

[R. Duan/AW, in preparation.]

$$\alpha^*(K) \leq A(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq S \leq \mathbb{1},$$

$$\operatorname{Tr}_A \Pi(S \otimes \mathbb{1})\Pi \leq \mathbb{1}.$$

(for cq-channels)

Thm. $C_{ONS}(K) = \lim \frac{1}{n} \log \alpha^*(K^{\otimes n}) = \log A(K)$.

[So, regularization of an SDP can itself be an SDP...

[R. Duan/AW, in preparation.]

$$\alpha^*(K) \le A(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \le S \le \mathbb{1},$$

$$\operatorname{Tr}_A \Pi(S \otimes \mathbb{1})\Pi \le \mathbb{1}.$$

(for cq-channels)

Thm. $C_{0NS}(K) = \lim \frac{1}{n} \log \alpha^*(K^{\otimes n}) = \log A(K).$

[So, regularization of an SDP can itself be an SDP... A(K) is "semidefinite packing number"; suggested by Aram Harrow. Has nice properties, e.g. multiplicativity.]

[R. Duan/AW, in preparation.]

$\alpha^*(K) \leq A(K) = \max \operatorname{Tr} S \text{ s.t. } 0 \leq S \leq \mathbb{1},$

$\operatorname{Tr}_A \Pi(S \otimes \mathbb{1})\Pi \leq \mathbb{1}.$

(for cq-channels)

Thm. $C_{ONS}(K) = \lim \frac{1}{n} \log \alpha^*(K^{\otimes n}) = \log A(K).$

[So, regularization of an SDP can itself be an SDP... $A(K)$ is "semidefinite packing number"; suggested by Aram Harrow. Has nice properties, e.g. multiplicativity.]

For two-pure-state channel: $A(K) = 1/\alpha^2$.

Summary (sort of)

$$\log \alpha^*(K) \leq C_{ONS}(K) \leq G_{ONS}(K) \leq \log \Sigma(K)$$

Summary (sort of)

$$\log \alpha^*(K) \leq C_{ONS}(K) \leq G_{ONS}(K) \leq \log \Sigma(K)$$

$$\leq -H_{min}(A|B)_J$$

Summary (sort of)

$$\log \alpha^*(K) \le C_{ONS}(K) \le G_{ONS}(K) \le \log \Sigma(K)$$

$$\le -H_{min}(A|B)_J$$

(can be strict)

$=/<?$

Summary (sort of)

$$\log \alpha^*(K) \leq C_{ONS}(K) \leq G_{ONS}(K) \leq \log \Sigma(K)$$

$$\leq -H_{min}(A|B)_J$$

(can be strict) $=/<$?

For cq-channel:

$$C_{ONS}(K) = \log \mathrm{A}(K) \geq \log \vartheta(G).$$

Summary (sort of)

$$\log \alpha^*(K) \leq C_{ONS}(K) \leq G_{ONS}(K) \leq \log \Sigma(K)$$

$$\leq -H_{min}(A|B)_J$$

(can be strict)

=/<?

For cq-channel:

$$C_{ONS}(K) = \log A(K) \geq \log \vartheta(G).$$

= attained for every G

Summary (sort of)

$$\log \alpha^*(K) \le C_{ONS}(K) \le G_{ONS}(K) \le \log \Sigma(K) \le -H_{min}(A|B)_J$$

(can be strict)

=/<?

For cq-channel:

$$C_{ONS}(K) = \log A(K) \ge \log \vartheta(G).$$

= attained for every G

First capacity interpretations of $\vartheta(G)$ & H_{min}