Universality of the Heisenberg limit for phase estimation

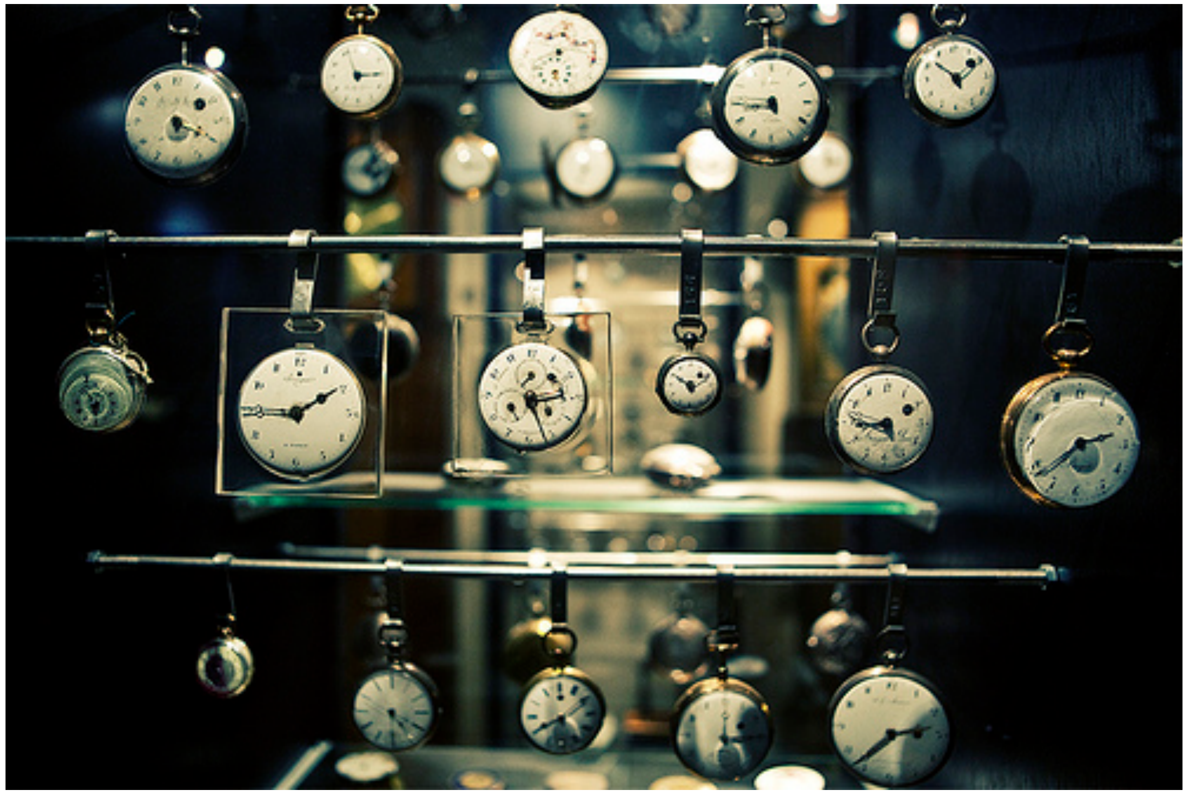
Marcin Zwierz, with Michael Hall,
Dominic Berry and Howard Wiseman

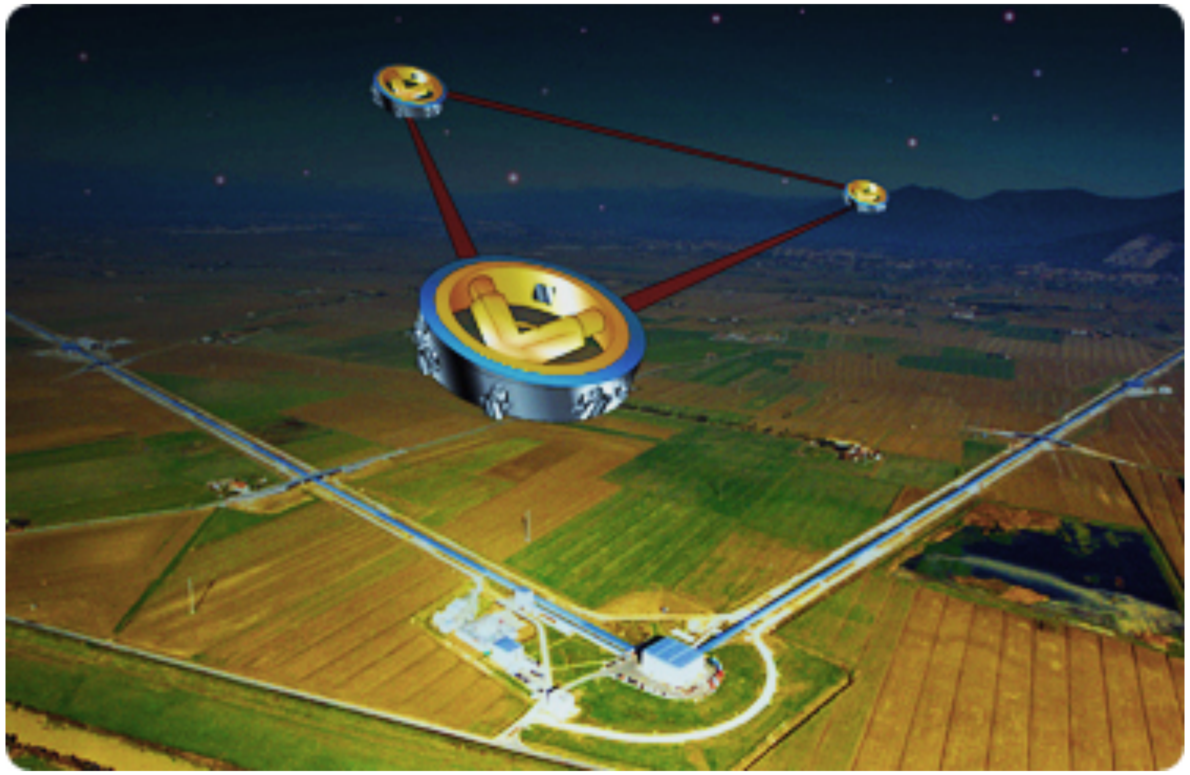
Centre for Quantum Dynamics
Griffith University
Australia

CEQIP 2013 Workshop

Why phase?


DEPARTMENT OF EDUCATION
RULEX-TRUELINE
COMMONWEALTH OF AUSTRALIA





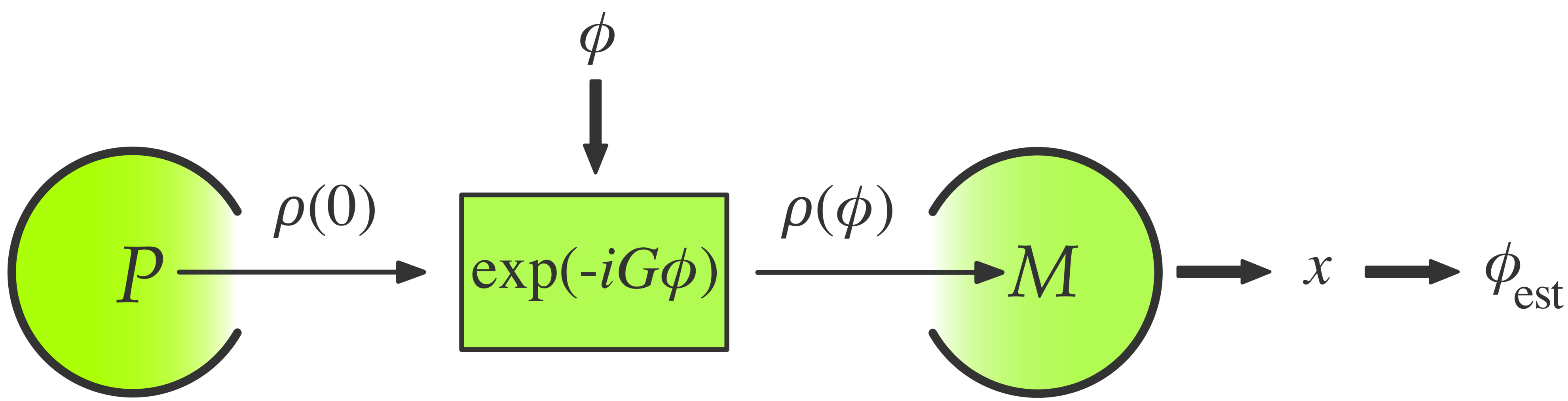
Outline

- Phase estimation
- Measures of phase resolution
- Quantum Cramér-Rao inequality and its limitations
- Generally accepted form of the Heisenberg limit
- Schemes with illusory improvements
- Universal Heisenberg limit
- Conclusions

M. J. W. Hall, D. W. Berry, M. Zwierz, and H. M. Wiseman, *Phys. Rev. A* **85**, 041802(R) (2012)

D. W. Berry, M. J. W. Hall, M. Zwierz, and H. M. Wiseman, *Phys. Rev. A* **86**, 053813 (2012)

Phase estimation



A probe system, described by some density operator ρ(0) undergoes a phase shift ϕ to become $\rho(\phi) = \exp(-i\phi G)\, \rho(0)\, \exp(i\phi G)$. Here G is some operator, for example

- $G = N$ or N^2 (photon number of a single-mode probe)
- $G = N_1 + N_2 + N_3 + \ldots$ (photon number of a multimode probe)

Then a generalized (possibly adaptive) measurement M is used to make an estimate, ϕ_{est}, of value of ϕ.

QUESTION: Is the estimate ϕ_{est} any good?

Measures of phase resolution

- Mean-square error:

$$\mathrm{MSE}_\phi := (\Delta_\phi\, \phi_{\mathrm{est}})^2 = \langle (\phi_{\mathrm{est}} - \phi)^2 \rangle_\phi$$

  Remark: MSE_ϕ is a measure of phase resolution only for a specific ϕ!

- Holevo variance:

$$V_{H,\phi}(\phi_{\mathrm{est}}) := |\langle e^{i\phi_{\mathrm{est}}} \rangle_\phi|^{-2} - 1$$

Quantum Cramér-Rao inequality

If the estimate is *unbiased* in the neighborhood of some specific phase shift ϕ, the square root of the mean-square error (MSE_ϕ) can be *locally* lower bounded with

$$\Delta_\phi \, \phi_{\text{est}} \geq \frac{1}{\sqrt{F_Q(\phi)}} \geq \frac{1}{2\Delta G}$$

Main limitation: the QCR bound holds only for *unbiased* estimates

$$\langle \phi_{\text{est}} \rangle_\phi = \phi$$

Unbiased estimates are very rare!

Heisenberg limit

A key concept in phase estimation is the Heisenberg limit

$$\sigma(\phi_{\text{est}}) \gtrsim k/\langle G \rangle$$

Remark: the Heisenberg limit is valid only for certain phase estimation schemes with single-mode probes but otherwise open to challenge!

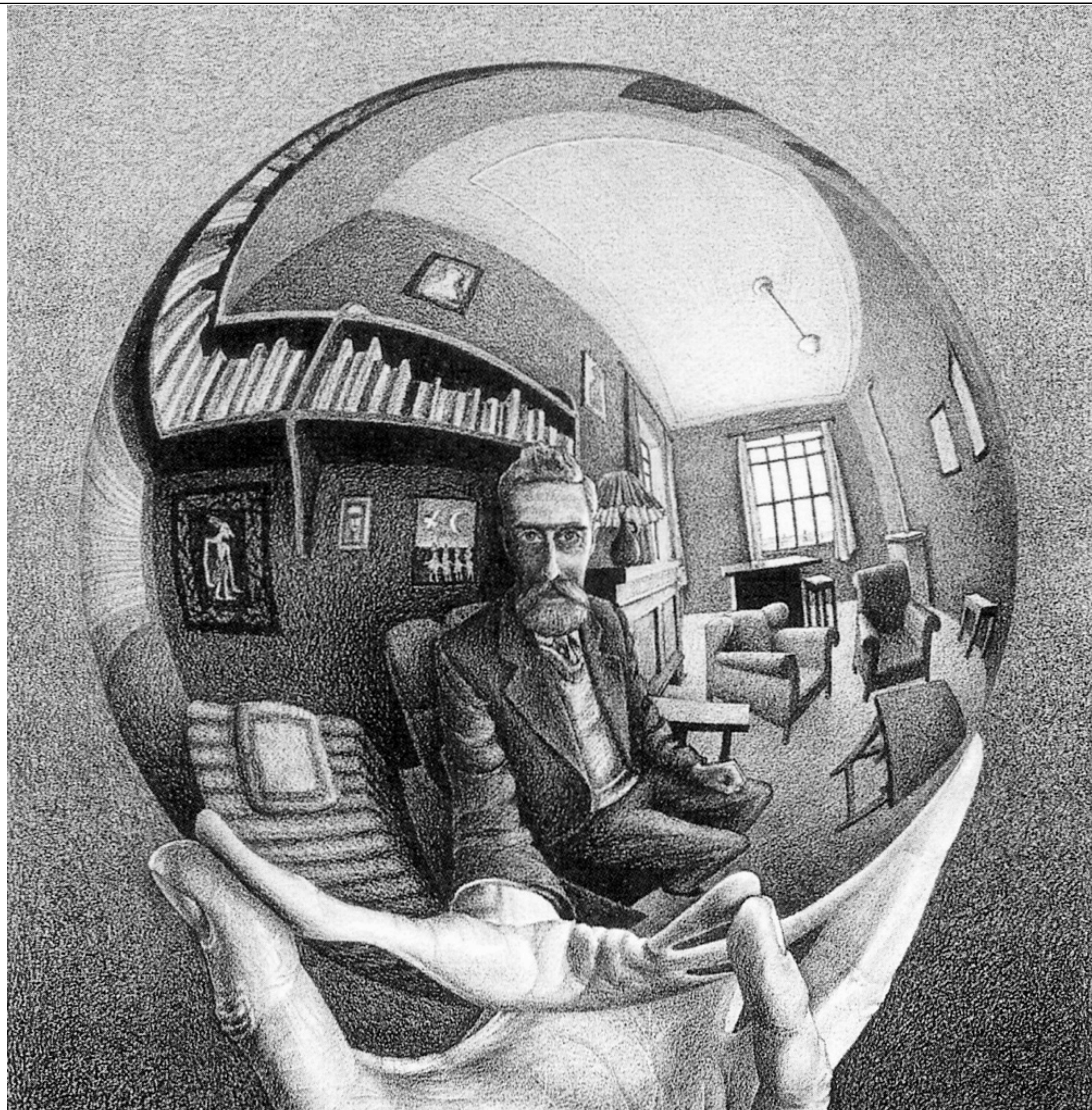
What about

- multimode fields?
- multiple passes of probe states?
- nonlinear phase shifts?
- special (noncovariant and/or entangling) measurements?

M. J. Holland and K. Burnett, *Phys. Rev. Lett.* **71,** 1355 (1993)

G. S. Summy and D. T. Pegg, *Opt. Commun.* **77**, 75 (1990).

Illusory improvements



Restricted range of phases

PRL **104,** 103602 (2010) PHYSICAL REVIEW LETTERS week ending 12 MARCH 2010

Quantum Metrology with Two-Mode Squeezed Vacuum: Parity Detection Beats the Heisenberg Limit

Petr M. Anisimov,* Gretchen M. Raterman, Aravind Chiruvelli, William N. Plick, Sean D. Huver, Hwang Lee, and Jonathan P. Dowling

Given $G = N$, they obtain

$$\Delta_\phi \, \phi_{\text{est}} = \frac{1}{\sqrt{\langle N \rangle (\langle N \rangle + 2)}}$$

Remark: up to a leading order this bound is linear in $\langle N \rangle$, therefore

- this result violates the Heisenberg limit only for small $\langle N \rangle$
- and only in a *small* range of phase shifts about $\phi = 0$

Restricted range of phases and bias

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

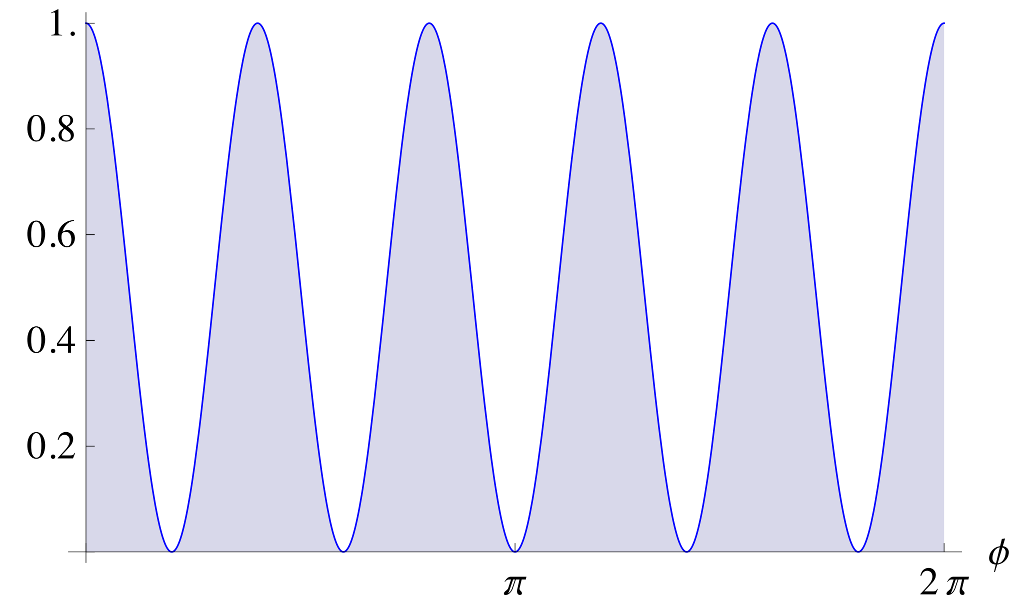
J. Phys. A: Math. Theor. **46** (2013) 035302 (10pp) doi:10.1088/1751-8113/46/3/035302

Unbounded quantum Fisher information in two-path interferometry with finite photon number

Y R Zhang[1], G R Jin[2], J P Cao[1], W M Liu[1] and H Fan[1]

$$|\psi\rangle = \sum_n \frac{c_n}{\sqrt{2}} \left[|n,0\rangle + |0,n\rangle\right]$$

$$\Delta_\phi \, \phi_{\text{est}} \geq 0$$



Remark: the authors use the quantum Cramér-Rao bound, but

- they use the QCRB for unbiased measurements
- a simple biased measurement can only yield zero error for $\phi = 0$ and π

Restricted range of phases

New Journal of Physics
The open-access journal for physics

Sub-Heisenberg estimation of non-random phase shifts

Ángel Rivas[1] **and Alfredo Luis**[2,3]

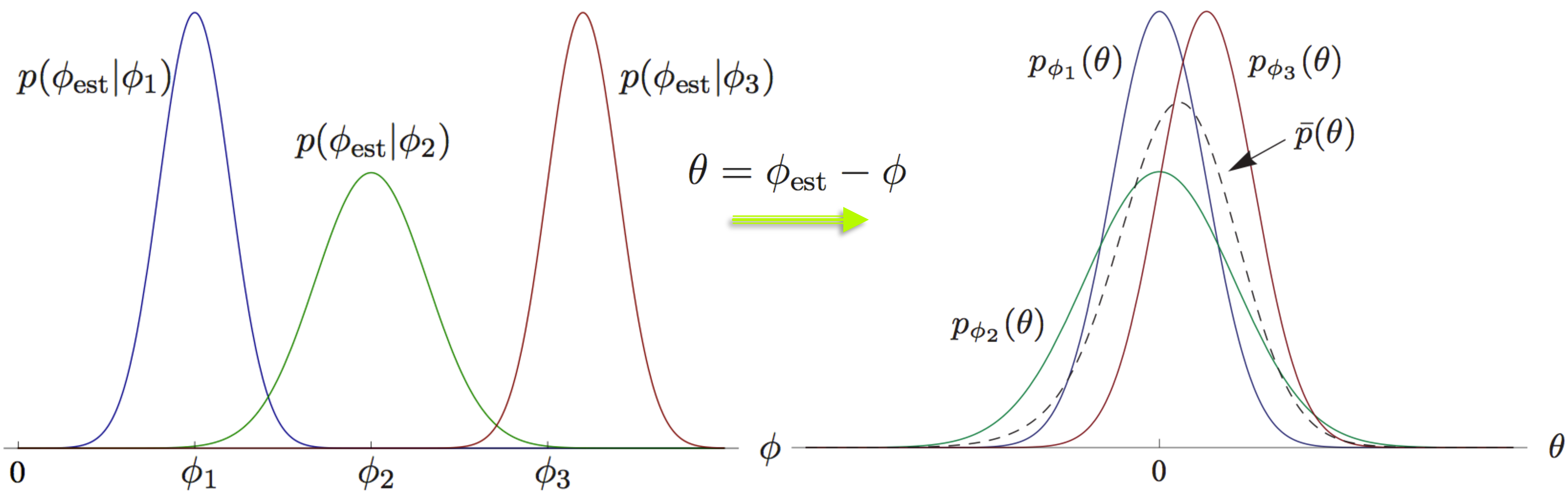
New Journal of Physics **14** (2012) 093052 (11pp)

$$\Delta_\phi\, \phi_{\text{est}} \propto \frac{1}{\langle N \rangle^p}$$

Remark: the authors use a coherent superposition of the vacuum and a squeezed state as their probe state, but again the phase shift that can be resolved is limited to small values: $\phi \ll 1$

The universal form of the Heisenberg limit?

Average performance



The overall performance of the estimate can be characterized by the concentration of the *average* probability distribution

$$\bar{p}(\theta) = \int_0^{2\pi} d\phi \, p(\phi) \, p(\theta + \phi|\phi)$$

We assume $p(\phi) = 1/(2\pi)$, that is, no prior information!

Average performance

A rigorous way of taking account of the range of phase is to average the mean-square error (MSE_ϕ) over *all* possible randomly applied phase shifts. Thus, the average mean-square error (AMSE) is defined by

$$(\delta\phi_{\mathrm{est}})^2 := \frac{1}{2\pi}\int_0^{2\pi} d\phi\,(\Delta_\phi\,\phi_{\mathrm{est}})^2$$

This holds because the mean-square error is a linear measure of phase resolution.

Universal Heisenberg limit

Lemma: Arbitrary phase measurement ↔ Canonical phase measurement

Given the entropic uncertainty relations for the canonical phase measurements on single-mode probes, we have

$$\delta\phi_{\mathrm{est}} > \frac{k_A}{\langle G+1 \rangle} \qquad \text{with} \qquad k_A := \sqrt{2\pi/e^3} \approx 0.559$$

No phase estimation scheme can do better, *on average*, than the Heisenberg limit!

Universal Heisenberg limit

This statement of the Heisenberg limit

$$\delta\phi_{\mathrm{est}} > \frac{k_A}{\langle G+1\rangle}$$

- ☑ is a non-asymptotic analytic lower bound that holds for all $\langle G\rangle$
- ☑ applies to all possible phase measurement schemes, and any estimate biased or unbiased of a completely random phase shifts; no prior information is available
- ☑ implies that the accuracy of any scheme violating the Heisenberg limit is essentially illusory

M. J. W. Hall, D. W. Berry, M. Zwierz, and H. M. Wiseman, *Phys. Rev. A* **85**, 041802(R) (2012)

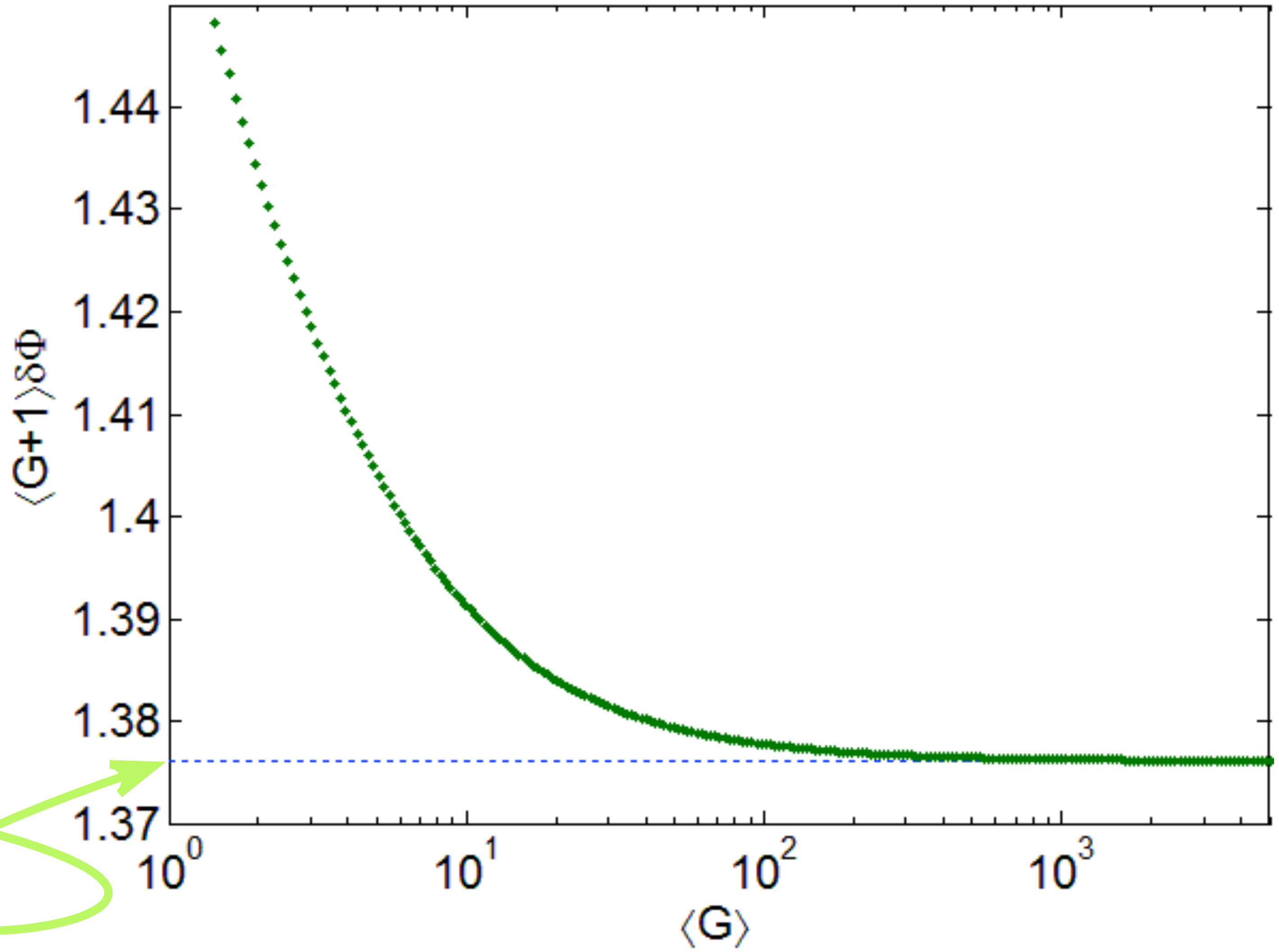
D. W. Berry, M. J. W. Hall, M. Zwierz, and H. M. Wiseman, *Phys. Rev. A* **86**, 053813 (2012)

Conjecture

The optimal lower bound:

$$\delta\phi_{\text{est}} > \frac{k_C}{\langle G+1 \rangle}$$

$$k_C := 2(-z_A/3)^{3/2} \approx 1.376$$



$$\delta\phi_{\text{est}} \gtrsim \delta_H\phi_{\text{est}} \gtrsim \frac{k_C}{\langle G \rangle}$$

D. W. Berry, M. J. W. Hall, M. Zwierz, and H. M. Wiseman, *Phys. Rev. A* **86**, 053813 (2012)

Conclusions

- We proved a general form of the Heisenberg limit for the average error of arbitrary phase measurements, provided that the phase is *a priori* completely unknown
    - ❖ the case where the prior information is available was addressed in M. J. W. Hall and H. M. Wiseman, *NJP* **14**, 033040 (2012)
- This result rules out the possibility of super-Heisenberg measurements
    - ❖ *local* super-Heisenberg measurements can be useful for phase sensing or phase tracking: H. Yonezawa *et al*., Science **337**, 1514-1517 (2012)

M. J. W. Hall, D. W. Berry, M. Zwierz, and H. M. Wiseman, *Phys. Rev. A* **85**, 041802(R) (2012)

D. W. Berry, M. J. W. Hall, M. Zwierz, and H. M. Wiseman, *Phys. Rev. A* **86**, 053813 (2012)

Thank you for your attention!

Also check out my poster on
Nonlinear quantum metrology with noise