LOCC transformations and the maximally entangled set of multipartite quantum states

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joint work with
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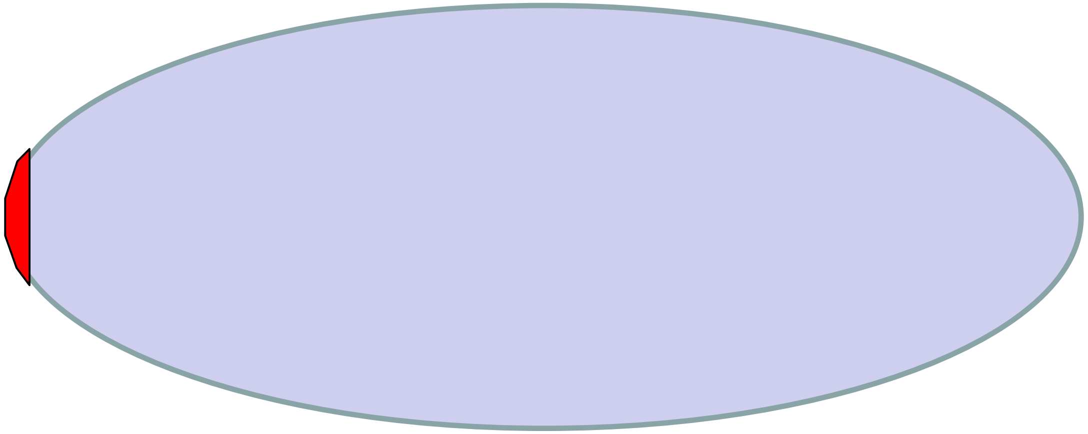
[JdV, Spee, Kraus, PRL 111, 110502 (2013)]

[Spee, JdV, Kraus, soon in arXiv]

Hilbert space is huge

Multipartite quantum states: # free parameters scales exponentially with the parties.



- Hard to grasp.
- Strong evidence that relevant states are of measure zero:
  - Few states actually used in applications, e.g. measurement-based q computation
  - Almost all states are useless for quantum computation [Gross et al., Bremner et al. PRL 09]
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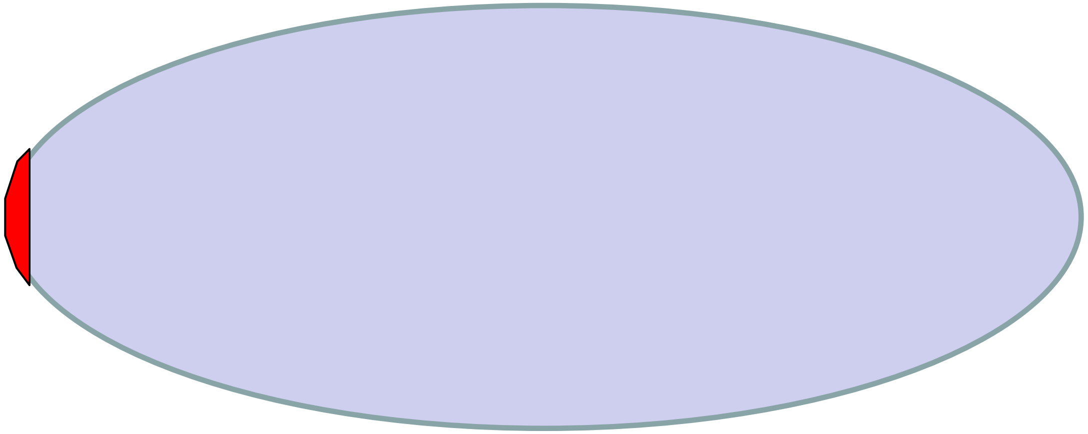

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e.g. Graph states, stabilizer states, LME states, M states, MPS, PEPS ...

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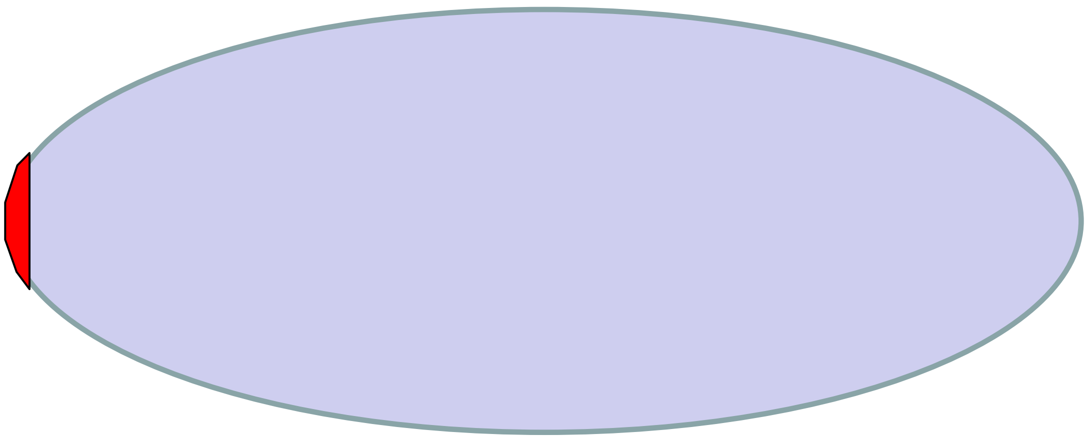

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We will study the "Maximally entangled set" (MES) of states

What is (not) maximal entanglement?

- Bipartite case clear: $|\psi\rangle \simeq_{LU} |\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$

but what about the multiqubit case? ⇨ Author dependent answer!

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  - only $\exists$ for 3,5,6 (7?) qubits! [Bravyi,PRA,03] [Scott,PRA,04]
  - Otherwise, maximize on average some measure of mixedness.

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Convertibility under deterministic LOCC provides the natural ordering among entangled states!

- $|\phi^+\rangle$ is ***the*** MES since it is the ***only*** state that can be transformed ***to any other*** by LOCC.
- Maximal entanglement (i.e. maximal usefulness) should correspond to the top state(s) according to this ordering.

Entanglement theory

- Characterization: Decide which states are entangled.
- Manipulation: Decide which tranformations are possible and provide protocols.
- Quantification: Order the set of states, measure how useful they are and how efficient manipulations can be.

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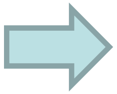

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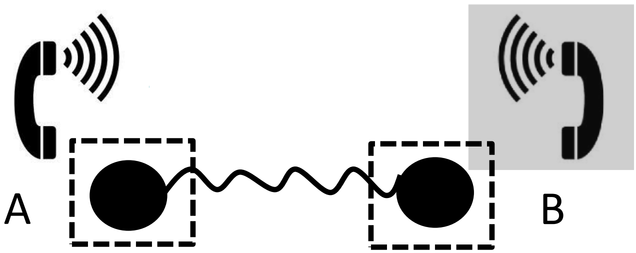
LOCC operations:



- Most general transformations achievable by spatially separated parties
  - Characterizes all possible protocols
- If $|\psi\rangle \rightarrow_{LOCC} |\phi\rangle$, then Ψ is not less useful than Φ
  - Induces an operational ordering in the set of entangled states

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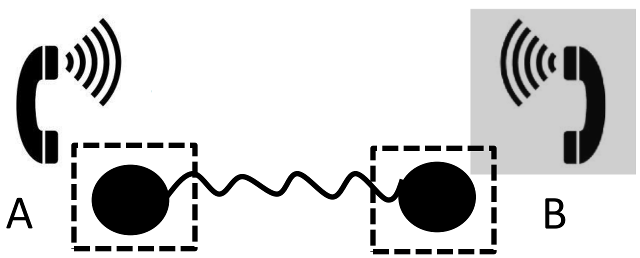

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Basic law of entanglement:
Entanglement cannot increase under LOCC

$$|\psi\rangle \rightarrow_{LOCC} |\phi\rangle$$
$$\Rightarrow E(\psi) \geq E(\phi)$$

Bipartite entanglement transformations

- Schmidt decomposition: $$|\psi\rangle_{12} \simeq_{LU} \sum_{i=1}^{d} \sqrt{\lambda_i}|i\rangle_1|i\rangle_2, \{\lambda_i\} = \mathrm{eig}(\rho_1)$$
- $\psi \longrightarrow_{LOCC} \phi$ iff $\sum_{j=1}^{k} \lambda_j^{\psi} \leq \sum_{j=1}^{k} \lambda_j^{\phi} \quad \forall k$ (majorization) [Nielsen PRL 1999]

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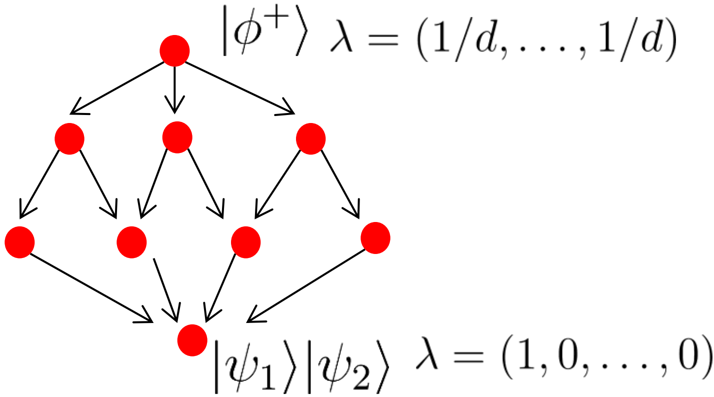

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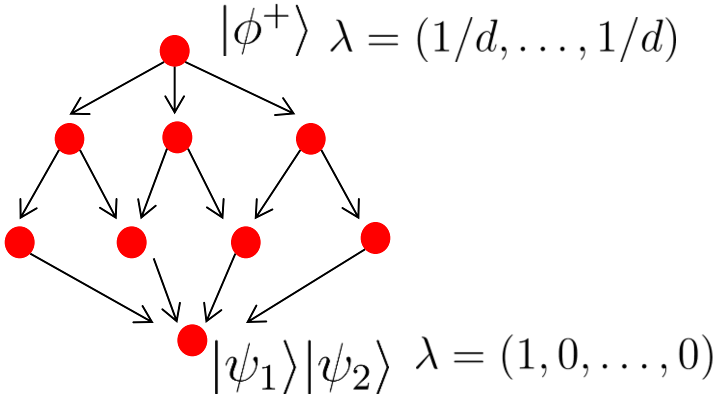

- $|\phi^+\rangle$ is the only state that cannot be obtained from any other.
- $|\phi^+\rangle$ is the only state that can be transformed to any other.

⇒ **Unique max. entgl. state**

⇒ **Maximal usefulness under LOCC conversions** $\Longrightarrow$ **Maximal entanglement**

Stochastic LOCC

- Characterization of deterministic LOCC transformations gets very complicated!
- Consider instead stochastic LOCC (SLOCC) convertibility.

Two states are in the same SLOCC class if they can be transformed into each other with a non-zero probability of success.

$$|\psi\rangle \simeq_{SLOCC} |\phi\rangle \Leftrightarrow |\phi\rangle = A \otimes B \otimes \cdots |\psi\rangle$$

$$\det(A) \neq \det(B) \neq \cdots \neq 0$$

[Dür,Vidal,Cirac, PRA, 2000]

2 SLOCC classses for 3 qubits:

$$|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$$

$$|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$$

⇒ There cannot be a unique maximally entangled state for 3-qubit states!

The maximally entangled set of n-partite states (n-MES)

A n-MES is the minimal set of n-partite states from which any other n-partite state can be obtained deterministically by LOCC.

- No state in n-MES can be obtained from any other by LOCC (excluding LUs).
- Every n-partite state which is not in n-MES can be obtained by LOCC by at least one state in n-MES.

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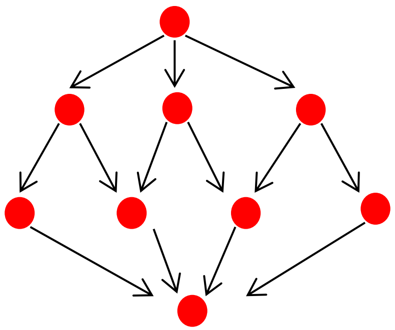
RESULTS:

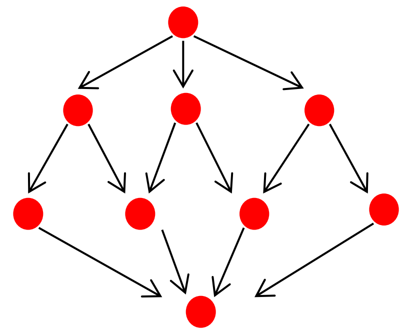
- **We show that multipartite LOCC transformations can be characterized.**
- **We determine the 3-qubit and 4-qubit MES.**

3-qubit MES

We could hope for something like:



GHZ



W

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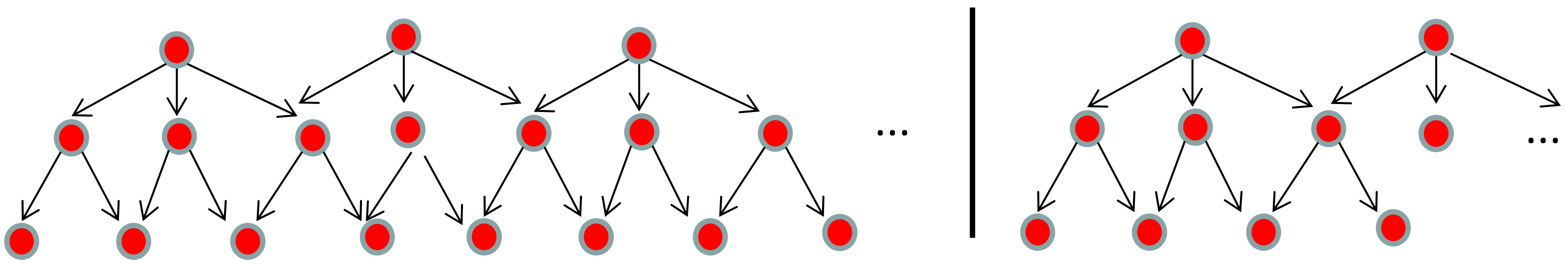

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3-qubit MES

See also:
[Turgut et al. PRA 2010]
[Kintas,Turgut, JMP 2010]

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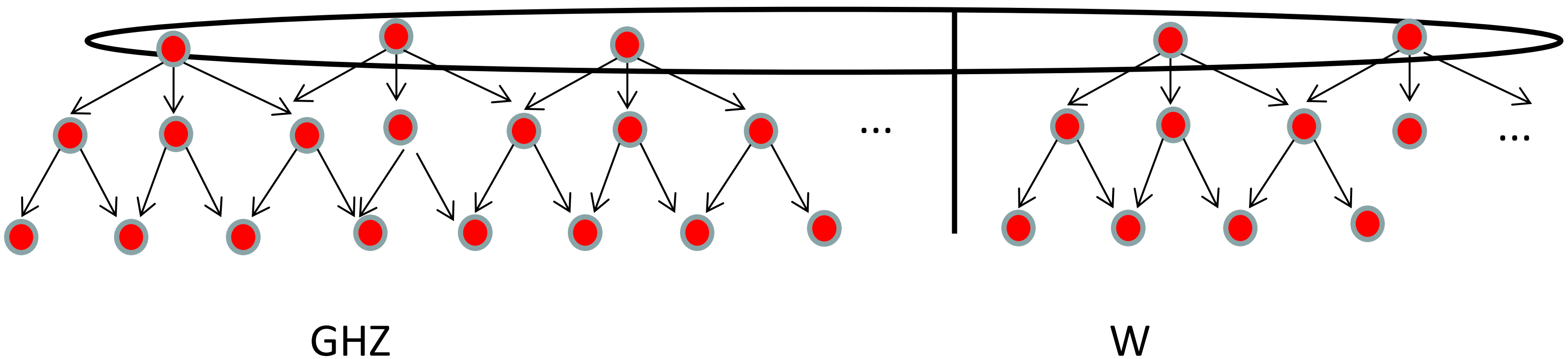
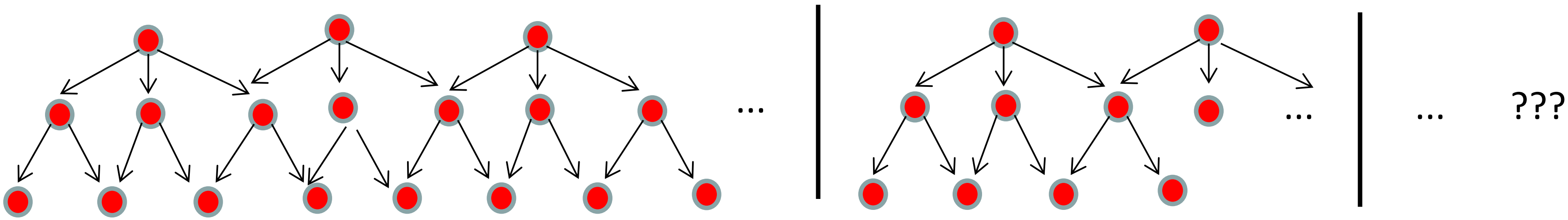


RESULTS:

- 3-qubit LU classes can be parametrized by 5 real parameters.
- Characterization of the MES corresponds to a 3-dimensional submanifold.
- **There are infinitely many MES but they form a zero-measure subset!**

4-qubit MES

There exist already infinitely many SLOCC classes [Verstraete et al. PRA 2002]



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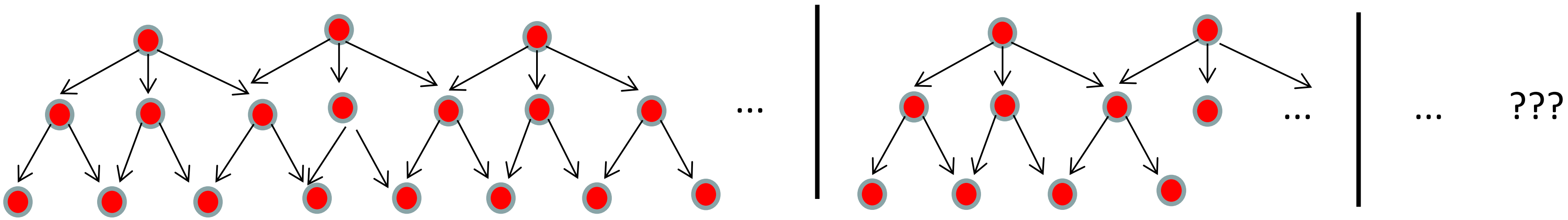

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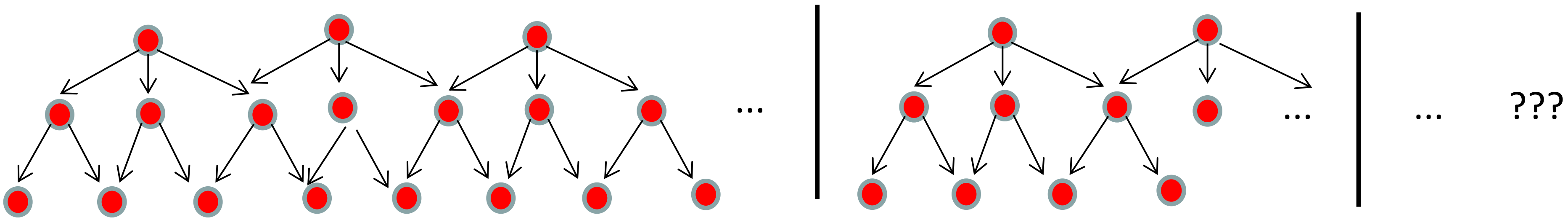

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- Almost all 4-qubit states are in the MES.

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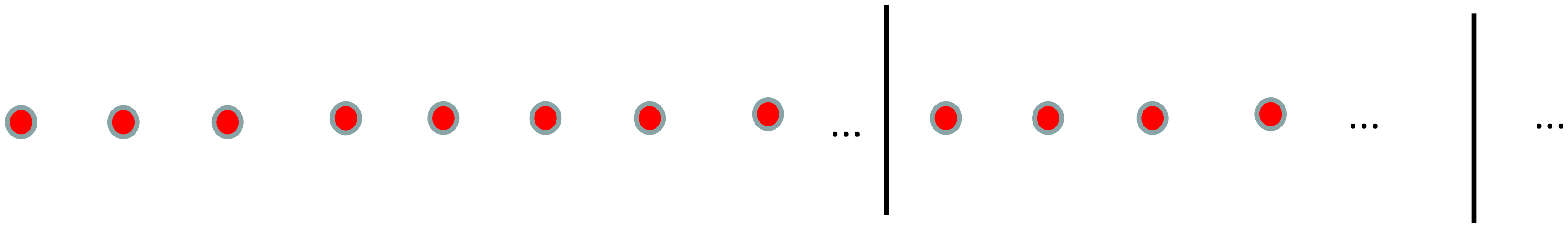

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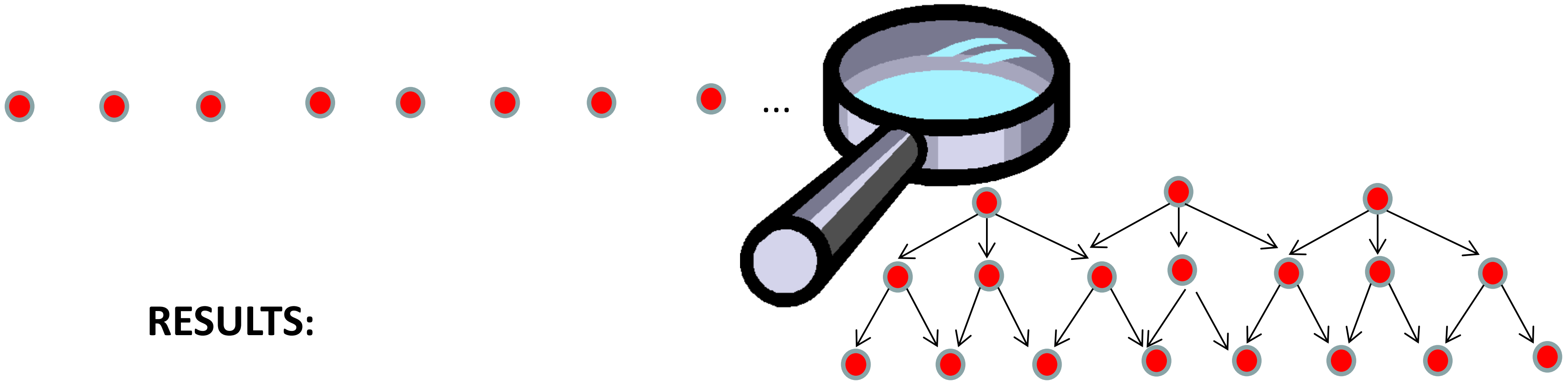


RESULTS:

- LOCC convertibility can be characterized and symmetry plays a fundamental role.
- Almost all 4-qubit states are in the MES.
- Almost all 4-qubit states are isolated (i.e. useless).
- Convertible (i.e. useful) states in the MES (zero-measure subset) are identified together with the possible LOCC transformations.

The method

1) Identify the different **SLOCC classes** together with a representative with a well defined mathematical structure that we will call **seed state** $|\psi_{seed}\rangle$.

2) Provide a **unique LU standard form** for all states in a given SLOCC class identifying all possible free parameters (analogous to Schmidt coefficients).

3) Characterize how these parameters can change under the mathematically simpler **separable transformations (SEP)**.

(since SEP$\subset$ LOCC -> necessary conditions for LOCC transformations)

4) The possible SEP transformations are already so limited that it turns out that they can always be implemented by very simple LOCC protocols (**1-way LOCC)**.

(sufficient conditions for LOCC transformations)

Example: 4-qubits G_{abcd} SLOCC classes

- A generic 4-qubit state can be written as $|\psi\rangle \propto A \otimes B \otimes C \otimes D|\psi_{seed}\rangle$, where

$$|\psi_{seed}\rangle = a(|0000> + |1111>) + b(|0011> + |1100>)$$

$$+c(|0101> + |1010>) + \mathrm{d}(|0110> + |1001>)$$

[Verstraete et al. PRA 2002]

➡ Generic 4-qubit SLOCC classes parametrized by 6 real **seed parameters**

- General states in a given SLOCC class parametrized by the seed parameters +

$$A \otimes B \otimes C \otimes D|\psi_{seed}\rangle$$

How to describe states in a given LU class uniquely?

➡ It can be shown that the 12 **SLOCC parameters** characterize generic states uniquely:

$$T^{\dagger}T = \frac{1}{2}(\mathbb{1} + t_x\sigma_x + t_y\sigma_y + t_z\sigma_z) \quad (T = A, B, C, D; t = a, b, c, d)$$

$$(0 \leq t_x^2 + t_y^2 + t_z^2 < 1 \quad \forall t)$$

Generic 4-qubit state MES

A 4-qubit generic state is in the MES and is LOCC convertible iff all SLOCC parameters vanish save at most 4 corresponding to the same direction.

E.g. $|\psi_{seed}\rangle, |\psi\rangle \propto A \otimes B \otimes C \otimes D|\psi_{seed}\rangle$ with non-zero a_x, b_x, c_x and d_x

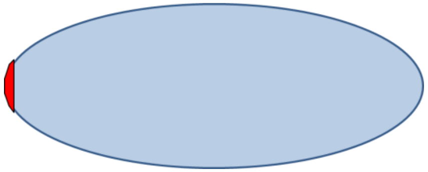
LOCC convertible states in the MES: 6 seed parameters + 4 SLOCC parameters = 10 parameters

(out of 18 total parameters)

Conclusions / future work

- We have introduced the notion of MES of states for the multipartite setting.
- Deterministic LOCC transformations among pure fully entangled multipartite states can be characterized. Symmetry plays a fundamental role.
- 3-qubit MES characterized. There are infinitely many states but it is of measure zero. LOCC conversions are always possible.
- 4-qubit MES characterized. Almost all states are in the MES. Almost all states are isolated.
  - ➢ LOCC convertibility induces a trivial ordering (most states are incomparable).
  - ➢ Almost all states are useless for entanglement conversions (i.e. they can only be deterministically transformed at the price of destroying all entanglement).
- Characterization of the zero-measure set of convertible states in the MES.

Identification of useful subclass of entangled states



Conclusions / future work

- Convertible MES $\Longrightarrow$ Operationally useful subclass of states

  Can one establish connenctions to other relevant subclasses/ find other applications?

- Extension of Nielsen theorem on the characterization of LOCC transformations to the multipartite case $\Longrightarrow$ Study new phenomena for the multipartite case:

  - Entanglement catalysis
  - Transformations on many copies
  - Optimal probabilities of LOCC transformations
  - Introduce new entanglement measures ...

- Extend our results to more parties and higher dimensions. Are almost all states isolated?

- Explore further the difference among SEP and LOCC transformations.