Quantum subdivision capacities and continuous quantum coding

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joint work with David Reeb and Michael M. Wolf

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TECHNISCHE UNIVERSITÄT MÜNCHEN

What is the optimal rate of information storage in a quantum memory?

Noise channel $\mathcal{T}_t : \mathfrak{M}_d \to \mathfrak{M}_d$ is Markovian, i.e. $\mathcal{T}_t = e^{t\mathcal{L}}$.

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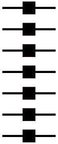

$T_t^{\otimes m}$

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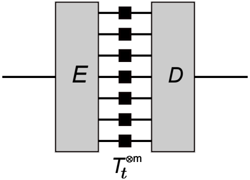

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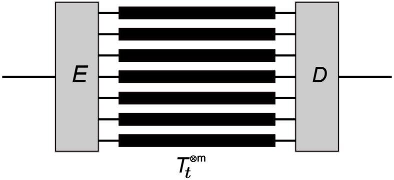

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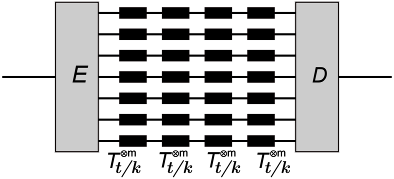

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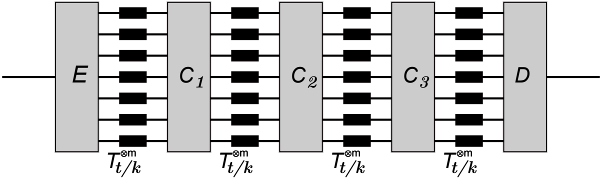

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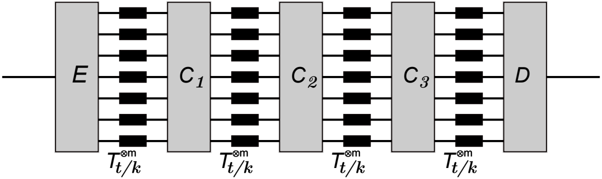

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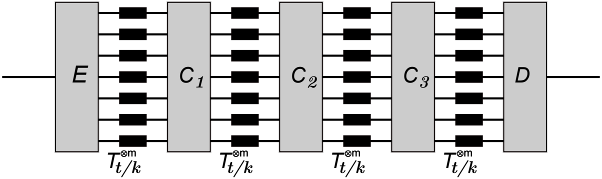

Quantum subdivision capacity: $\mathcal{Q}_{\mathfrak{C}}(t\mathcal{L})$

Different choices of $\mathfrak{C}$ lead to different capacities!

Definition (MH, Reeb, Wolf 2013)

The $\mathfrak{C}-$**quantum subdivision capacity** of $t\mathcal{L}$ is then defined as the supremum of asymptotical achievable rates

$$\mathcal{Q}_{\mathfrak{C}}(t\mathcal{L}) := \sup\{R \in \mathbb{R}^+ : \; R = \limsup_{\nu\to\infty} \frac{n_\nu}{m_\nu}\}.$$

such that the asymptotic communication error vanishes

$$\inf_{k,\mathcal{E},\mathcal{D},\mathcal{C}_1,\ldots,\mathcal{C}_k} \left\| \mathrm{id}_2^{\otimes n_\nu} - \mathcal{D} \circ \prod_{l=1}^{k} \left(\mathcal{C}_l \circ \left(e^{\frac{t}{k}\mathcal{L}} \right)^{\otimes m_\nu} \right) \circ \mathcal{E} \right\|_\diamond \to 0 \quad \text{as } \nu \to \infty.$$

Infimum goes over:

- $k \in \mathbb{N}$ number of subdivisions
- $\mathcal{E} : \mathfrak{M}_2^{\otimes n_\nu} \to \mathfrak{M}_d^{\otimes m_\nu}$ and $\mathcal{D} : \mathfrak{M}_d^{\otimes m_\nu} \to \mathfrak{M}_2^{\otimes n_\nu}$ quantum channels
- $\mathcal{C}_l \in \mathfrak{C}$ channels from the subset $\mathfrak{C}$

Outline

1. Infinitely divisible coding maps

2. Unitary coding maps

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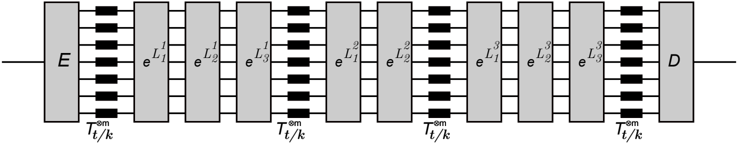

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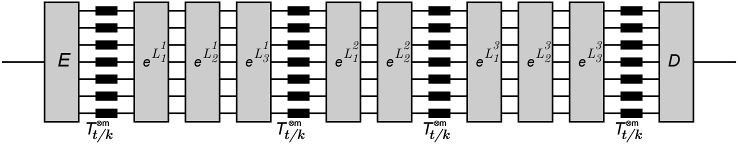

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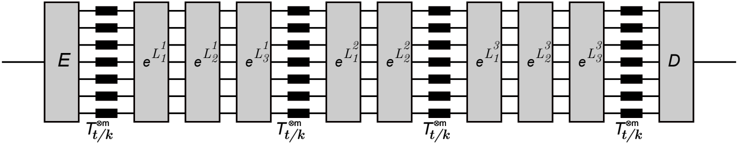

Theorem (MH, Reeb, Wolf 2013)

For any noise Liouvillian $\mathcal{L} : \mathfrak{M}_d \to \mathfrak{M}_d$ and any $t \in \mathbb{R}^+$ we have

$$\mathcal{Q}_{ID}(t\mathcal{L}) = \log(d)$$

Proof of $\mathcal{Q}_{\mathbf{ID}}(t\mathcal{L}) = \log(d)$

Continuity of quantum capacity:

$$\mathcal{Q}\left(e^{\frac{t}{k}\mathcal{L}}\right) \rightarrow \log(d), \quad k \rightarrow \infty$$

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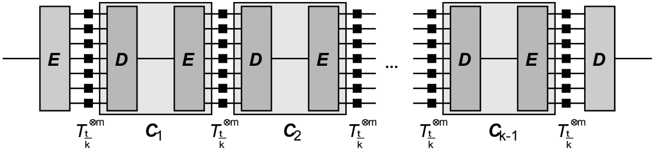

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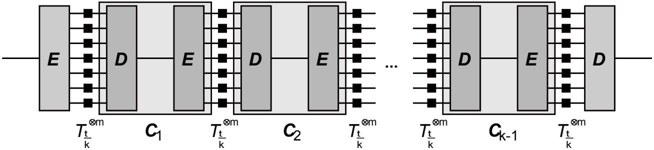

But the channel $\mathcal{D} \circ \mathcal{E}$ is not necessarily infinitely divisible!

Proof of $\mathcal{Q}_{\mathrm{ID}}(t\mathcal{L}) = \log(d)$

Implement intermediate coding via infinitely divisible channels

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Almost pure ancillas from the infinitely divisible channel:

$$\rho \mapsto (1 - e^{-rt})\mathrm{tr}(\rho) |0\rangle \langle 0| + e^{-rt}\rho \ , \quad \text{for large rate r}$$

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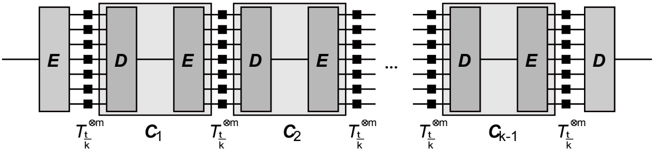

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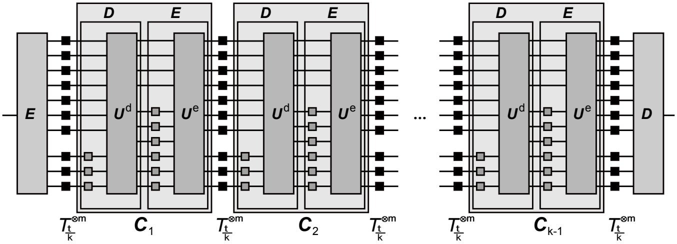

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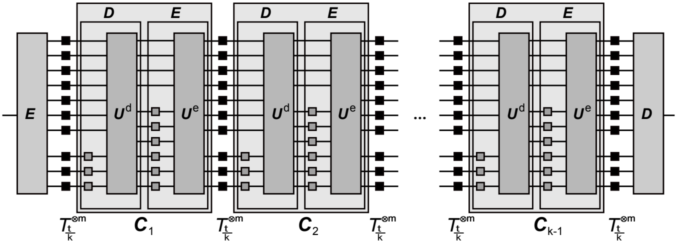
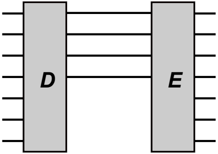


How many pure ancillas are sufficient?

Proof of $\mathcal{Q}_{\mathbf{ID}}(t\mathcal{L}) = \log(d)$

Decoupling approach to the quantum capacity:

$$\frac{1}{4}\inf_{\mathcal{D}}\left\|\mathrm{id} - \mathcal{D}\circ\mathcal{T}^{\otimes m}\circ\mathcal{E}\right\|_{\diamond}^{2} \leq \left\|\left(\mathcal{T}^{\otimes m}\circ\mathcal{E}\right)^{c} - \mathrm{tr}(\bullet)\,\sigma^{E}\right\|_{\diamond}$$



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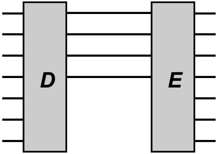

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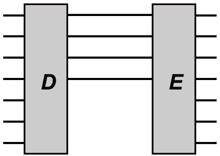

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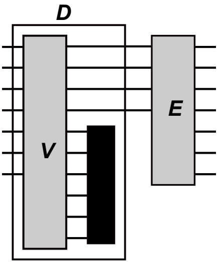

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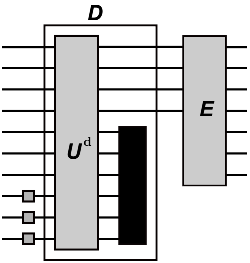

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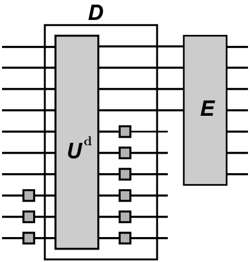

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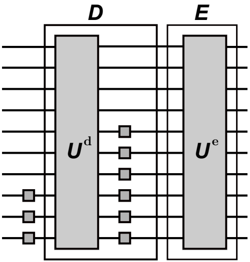

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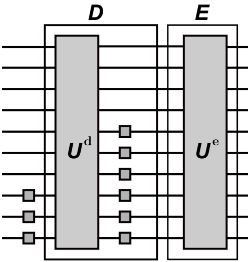

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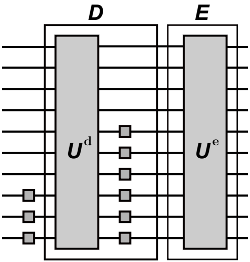

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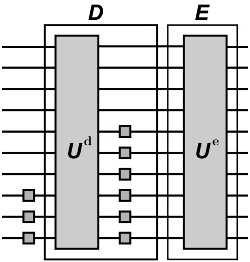

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$\rightsquigarrow$ **number of qubit ancillas sublinear in m**

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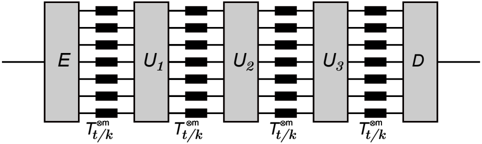

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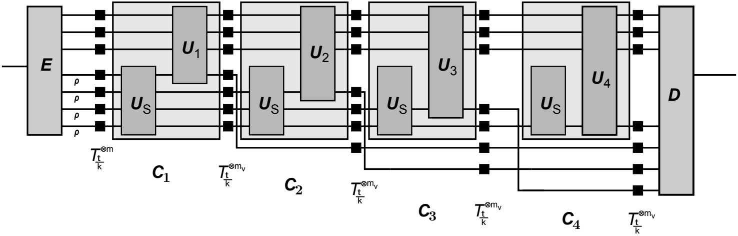

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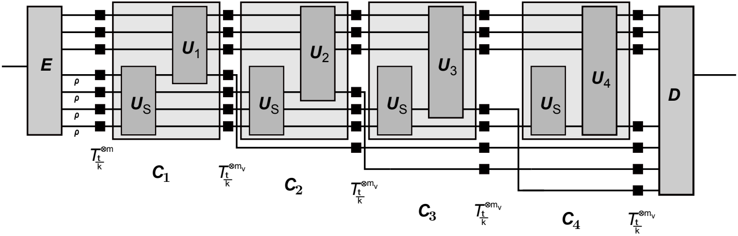

Theorem (MH, Reeb, Wolf 2013)

Let $\mathcal{L} : \mathfrak{M}_d \to \mathfrak{M}_d$ denote a Liouvillian with fixed point $\rho_0 \in \mathfrak{D}_d$. Then we have

$$\mathcal{Q}_{\mathrm{un}}(t\mathcal{L}) \geq \max_k \left(\frac{I^{coh}\left(\mathcal{T}_{t/k}\right)}{1 + 5k^{\frac{\log(d) - I^{coh}\left(\mathcal{T}_{t/k}\right)}{\log(d) - S(\rho_0)}}} \right) > 0.$$

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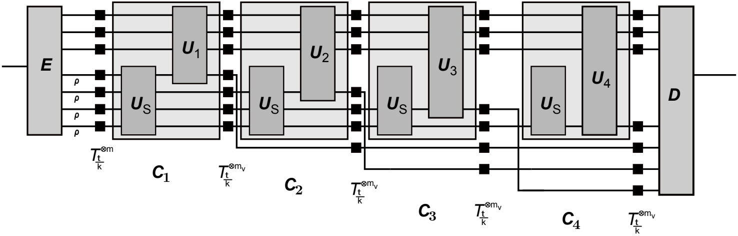

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$I^{\text{coh}}\left(\mathcal{T}_{t/k}\right) =$ **const. for** $k \sim t$. $\rightsquigarrow$ **Lower bound** $\sim \frac{1}{t}$

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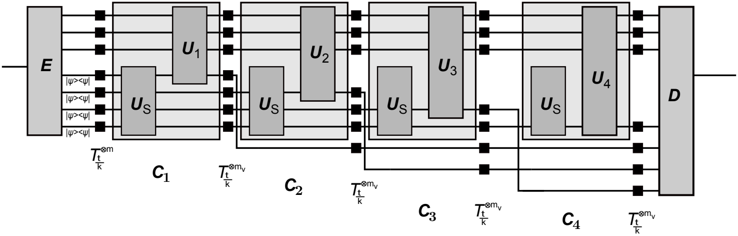

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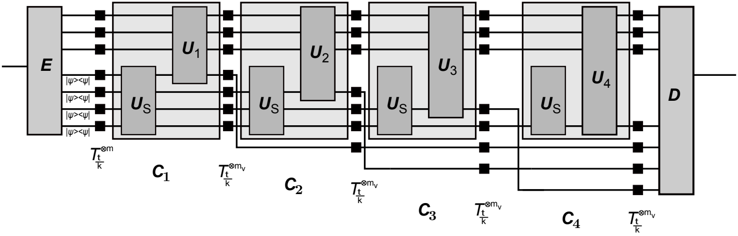

Maximal entropy of ancilla states never reached in finite time!

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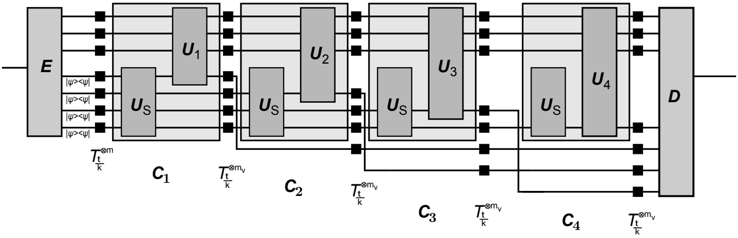

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Theorem (MH, Reeb, Wolf 2013)

Let $\mathcal{L} : \mathfrak{M}_d \to \mathfrak{M}_d$ denote a unital Liouvillian. Then we have

$$\mathcal{Q}_{\mathrm{un}}(t\mathcal{L}) > 0.$$

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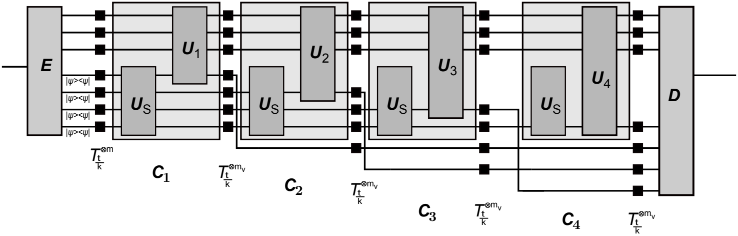

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$$\mathcal{Q}_{\mathrm{un}}\left(t\mathcal{L}\right) > 0.$$

$\rightsquigarrow$ $\mathcal{Q}_{\mathrm{un}}$ **is always strictly larger than zero.**

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Theorem

For any noise Liouvillian $\mathcal{L} : \mathfrak{M}_d \to \mathfrak{M}_d$ fulfilling $(\star)$ and $t \in \mathbb{R}^+$ we have

$$\mathcal{Q}_{\mathfrak{U}}(t\mathcal{L}) \leq e^{-2\alpha t}\log(d).$$

Proof of $\mathcal{Q}_{\mathfrak{U}}\left(t\mathcal{L}^{\mathbf{dep}}\right) \leq e^{-2\alpha t}\log(d)$

Consider subdivision coding scheme achieving rate R:

$$\inf_{k,\mathcal{E},\mathcal{D},\mathcal{C}_1,\ldots,\mathcal{C}_k} \left\| \mathrm{id}_2^{\otimes Rm} - \mathcal{D} \circ \prod_{l=1}^{k} \left(\mathcal{U}_l \circ \left(e^{\frac{t}{k}\mathcal{L}} \right)^{\otimes m} \right) \circ \mathcal{E} \right\|_{\diamond} \to 0$$

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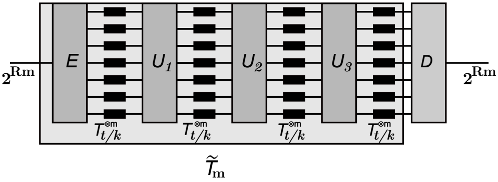

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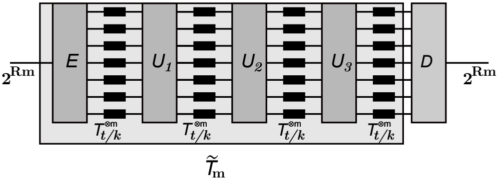

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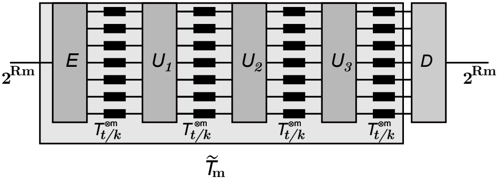

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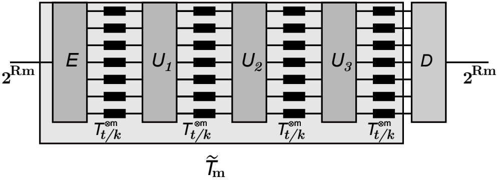

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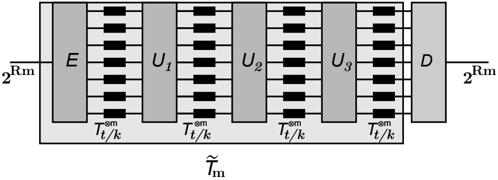

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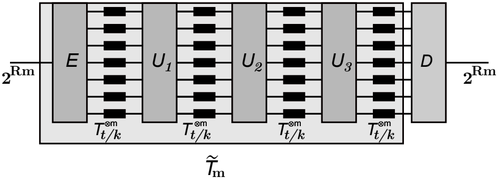

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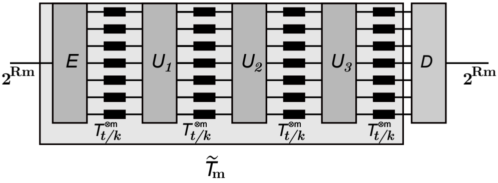

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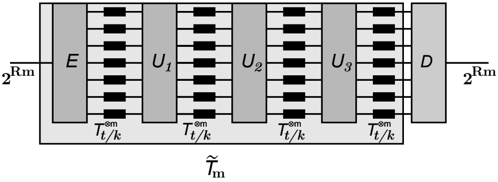

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$$D\left(\left(e^{t\mathcal{L}}\right)^{\otimes m}(\rho) \,\|\, \frac{\mathbb{1}_m}{d^m}\right) \leq e^{-2\alpha t} D\left(\rho \,\|\, \frac{\mathbb{1}_m}{d^m}\right) \qquad (\star)$$

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- **Depolarizing Liouvillian:** $\mathcal{L}^{\text{dep}}(\rho) := \lambda\left(\text{tr}(\rho)\frac{\mathbb{1}}{d} - \rho\right) \rightsquigarrow \alpha = \frac{\lambda}{2}$
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Shows that $\mathcal{Q}_{\mathfrak{U}}(t\mathcal{L}) \leq e^{-2\alpha t}\log(d)$ for any unital, detailed-balanced Liouvillian with unique fixed point.

What about non-unital Liouvillians?

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For more information see: arXiv:1310.2856