THE ULTIMATE LIMITS OF QUANTUM POSTSELECTION

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joint works with
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Deterministic vs probabilistic

Deterministic world

- Non-orthogonal states cannot be distinguished / cloned
- Coherent light cannot be amplified
- SQL for phase estimation / reference frame alignment with multiple copies

...

Probabilistic world

- Unambiguous state discrimination / cloning of linearly independent states (Duan-Guo 98)
- Noiseless amplifiers (Ralph-Lund 08)
- HL for phase estimation / reference frame alignment (Fiurásek 06, Bagan et al 12, Chiribella-Yang-Yao 13)

...

This talk

Ultimate limits and basic laws of probabilistic processes:

- Limits and benchmarks on probabilistic amplifiers
- Limits to the replication of quantum states
- Super-replication of states and gates

APPETIZER:

EXPLORING THE LIMITS
OF
PROBABILISTIC AMPLIFIERS

Amplifying coherent states of light

Coherent state:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$\alpha \in \mathbb{C}$

Ideally we wish to transform $|\alpha\rangle$ into $|g\alpha\rangle,\ g > 1$ ("amplifier gain")

$\longrightarrow$

No perfect amplification

The transformation $|\alpha\rangle \to |g\alpha\rangle \qquad \forall \alpha \in \mathbb{C}$
is not physically realizable.

For good reasons:

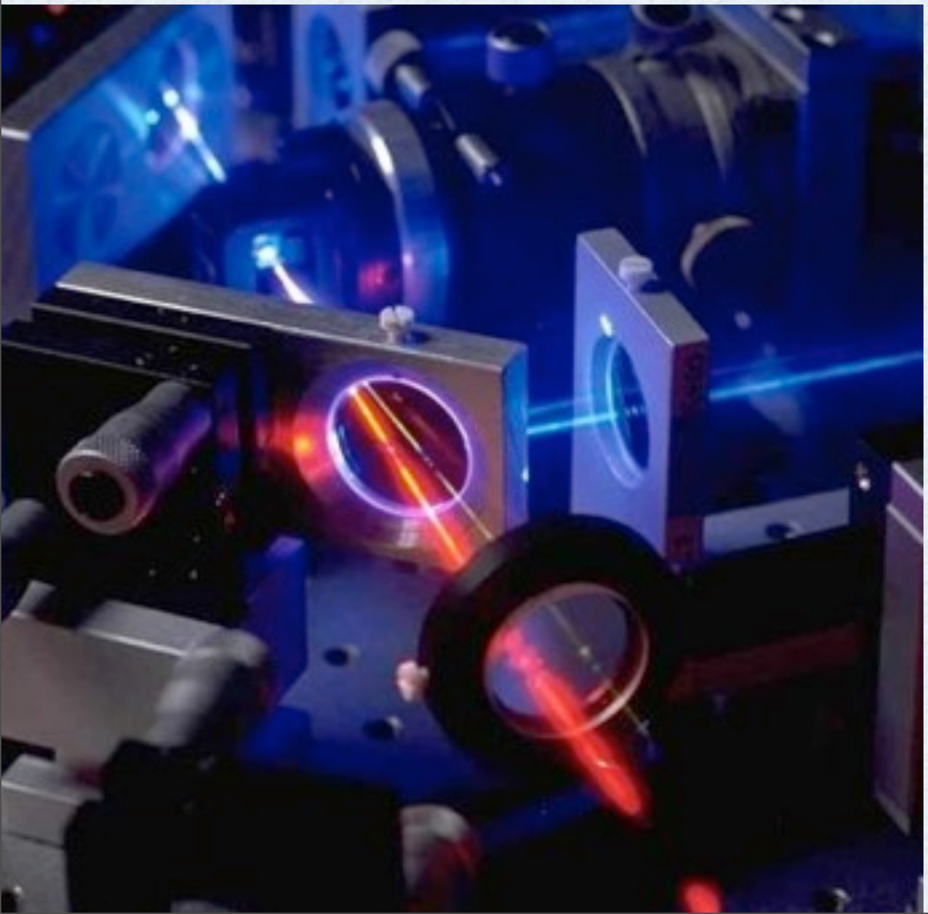
- it would violate Heisenberg's uncertainty principle
- it would lead to faster-than-light communication
- it would violate the no-cloning theorem

How can we approximate amplification with a physical process allowed by quantum mechanics?

Approximate deterministic amplification

Parametric amplifier

$$\mathcal{C}_r(\rho) = \mathrm{Tr}_B[\underbrace{e^{r(a^\dagger b^\dagger - ab)}}_{\text{two-mode squeezing operator}}(\rho \otimes \underbrace{|0\rangle\langle 0|}_{\text{ancillary mode in the vacuum state}})e^{-r(a^\dagger b^\dagger - ab)}]$$



For the input $|\alpha\rangle$

the output is a thermal state displaced by

$g\alpha, \qquad g = \cosh r$

Optimal for suitably chosen values of r
(Namiki PRA 2011,
Chiribella and Xie PRL 2013).

Noiseless probabilistic amplifiers

Ralph and Lund (2008) propose a probabilistic scheme achieving almost perfect amplification.

$$\mathcal{Q}_N(\rho) = Q_N\,\rho\,Q_N^\dagger \qquad Q_N := \sum_{n=0}^{N} \frac{g^n}{g^N}\,|n\rangle\langle n|$$

For large N: $\mathcal{Q}_N(|\alpha\rangle\langle\alpha|) \approx |g\alpha\rangle\langle g\alpha|$

(while the probability drops exponentially)

Probabilistic amplifiers in the lab

PRL **104**, 123603 (2010) PHYSICAL REVIEW LETTERS week ending 26 MARCH 2010

Implementation of a Nondeterministic Optical Noiseless Amplifier

Franck Ferreyrol, Marco Barbieri, Rémi Blandino, Simon Fossier, Rosa Tualle-Brouri, and Philippe Grangier

LETTERS
PUBLISHED ONLINE: 28 MARCH 2010 | DOI: 10.1038/NPHOTON.2010.35
nature photonics

Heralded noiseless linear amplification and distillation of entanglement

G. Y. Xiang[1], T. C. Ralph[2], A. P. Lund[1,2], N. Walk[2] and G. J. Pryde[1]*

nature physics
LETTERS
PUBLISHED ONLINE: 11 NOVEMBER 2012 | DOI: 10.1038/NPHYS2469

Heralded noiseless amplification of a photon polarization qubit

S. Kocsis[1,2], G. Y. Xiang[2,3], T. C. Ralph[1,4] and G. J. Pryde[1,2]*

nature physics
LETTERS
PUBLISHED ONLINE: 15 AUGUST 2010 | DOI: 10.1038/NPHYS1743

Noise-powered probabilistic concentration of phase information

Mario A. Usuga[1,2†], Christian R. Müller[1,3†], Christoffer Wittmann[1,3], Petr Marek[4], Radim Filip[4], Christoph Marquardt[1,3], Gerd Leuchs[1,3] and Ulrik L. Andersen[2]*

IOPscience
New Journal of Physics > Volume 15 > September 2013

A complete characterization of the heralded noiseless amplification of photons

OPEN ACCESS

N Bruno, V Pini, A Martin and R T Thew[1]

ARTICLES
PUBLISHED ONLINE: 21 NOVEMBER 2010 | DOI: 10.1038/NPHOTON.2010.260
nature photonics

A high-fidelity noiseless amplifier for quantum light states

A. Zavatta[1,2], J. Fiurášek[3] and M. Bellini[1,2]*

Questions

- How do these experiments compare with amplification strategies based on probabilistic estimation?
  (i.e. what is the relation between amplification and estimation inside the probabilistic world?)

- Is RL08 the optimal probabilistic process for amplification? Or is there an even better amplifier?

Modeling the source

To model the source of coherent states we assume a Gaussian distribution:

$$p_\lambda(\alpha) = \lambda e^{-\lambda|\alpha|^2}$$

with this choice the expected photon number is $\langle n \rangle = 1/\lambda$

λ represents our prior information about the input:

$$\lambda = 0 \quad \Rightarrow \quad \text{no information}$$

$$\lambda = \infty \quad \Rightarrow \quad \text{complete information}$$

The best probabilistic amplifiers

When the prior information is larger than a critical value,
nearly perfect amplification becomes possible!

$$F^{prob}_{g,\lambda} = \begin{cases} \dfrac{\lambda+1}{g^2}, & \lambda \leq g^2 - 1 \\ 1 & \lambda > g^2 - 1. \end{cases}$$

critical behavior at the value $\lambda_c^{prob} = g^2 - 1$

Above the critical value,
the best probabilistic amplifier becomes non-Gaussian and is achieved by RL08 protocol (Chiribella & Xie PRL 2013)
Having a non-flat prior is essential.

The best estimation-based amplifier

The best amplifier based on measurement and re-preparation is:

measurement: heterodyne POVM $$P_\alpha \, \mathrm{d}^2\alpha = |\alpha\rangle\langle\alpha| \, \frac{\mathrm{d}^2\alpha}{\pi}$$

states re-prepared for outcome α : $$\left| \frac{g}{1+\lambda} \, \alpha \right\rangle$$

Its fidelity is $$\widetilde{F}^{opt}_{g,\lambda} = \frac{1+\lambda}{1+\lambda+g^2}$$ (Chiribella & Xie PRL 2013)

Equal to the fidelity of the best deterministic protocol (!)
by Namiki, Koashi, Imoto, PRL 08

Application

Experiment designed to demonstrate high-fidelity probabilistic amplification with gain g = 2.

Values tested in the experiment:
$|\alpha| \approx 0.4/0.7/1.0$

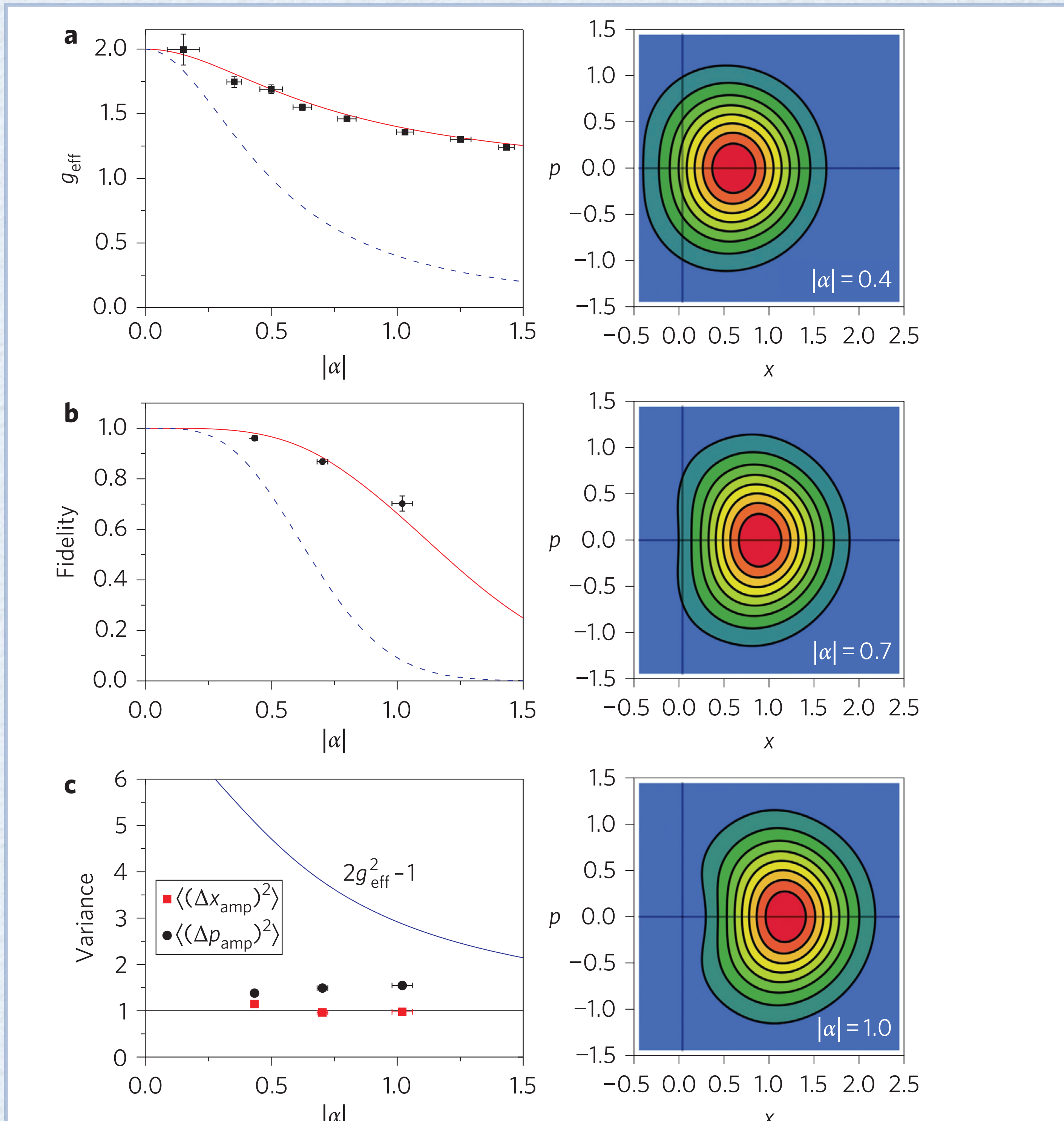
Experimental fidelities:
$F_{exp} \approx 0.99/0.91/0.67$

Reasonable choice of λ: $\lambda = 3$

gives the quantum benchmark

$$\widetilde{F}_{g=2,\lambda=3} = 50\%$$

passed by the experiment (although more data would be needed for a conclusive assessment)



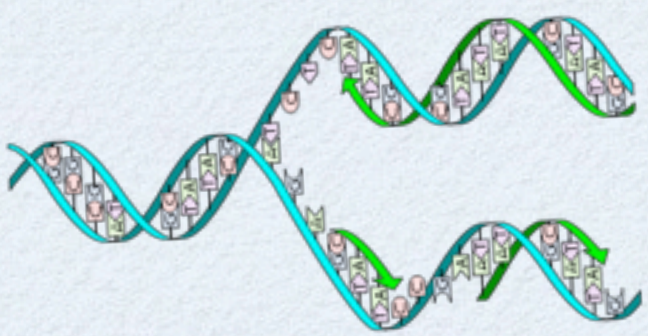
from Zavatta, Bellini, and Fiurasek, Nature Photonics 2010

MAIN COURSE:

QUANTUM SUPER-REPLICATION

Copying data

Copying is a fundamental task,
with applications across the most diverse fields,
including cryptography, market and technology, biology and art.



In the quantum world, the no-cloning theorem forbids perfectl replication.
But what are the limits to approximate replication?

Quantum replication: basic definitions

Replication process: transforms N perfect copies into M = M (N) approximate copies:

$$|\psi_\theta\rangle^{\otimes N} \mapsto |\Psi'_\theta\rangle \approx |\psi_\theta\rangle^{\otimes M} \qquad \theta \in \Theta, M \geq N$$

Quality of the copies measured by the global fidelity:

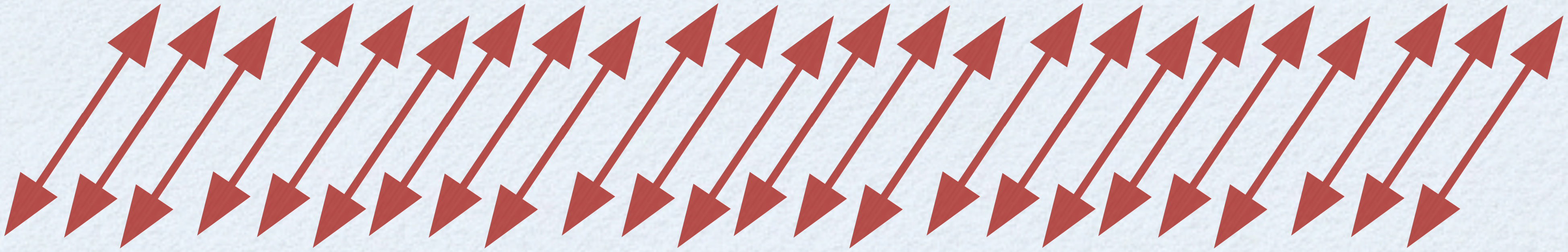
$$F_{N\to M(N)} = \int \mathrm{d}\theta \; p(\theta) \; \left|\langle\Psi'_\theta|\psi_\theta\rangle^{\otimes M(N)}\right|^2$$

Reliable replication: $\lim_{N\to\infty} F_{N\to M(N)} = 1$

Replication rates

Replication rate: $\delta N \approx const \times N^{\alpha}$ $\delta N = M(N) - N$

e. g. linear rate $\alpha = 1$



A rate is achievable iff exists a reliable replication process with that rate

Replication capacity

Replication capacity of a set of states:

$$\alpha^* = \sup\{\alpha : \ \alpha \text{ is an achievable replication rate}\}$$

Deterministic replication

For arbitrary qubit states: $\alpha^* = 1$

For qubit states on the equator of the Bloch sphere: $\alpha^* = 1$

For arbitrary qudit states: $\alpha^* = 1$

...

WHY?

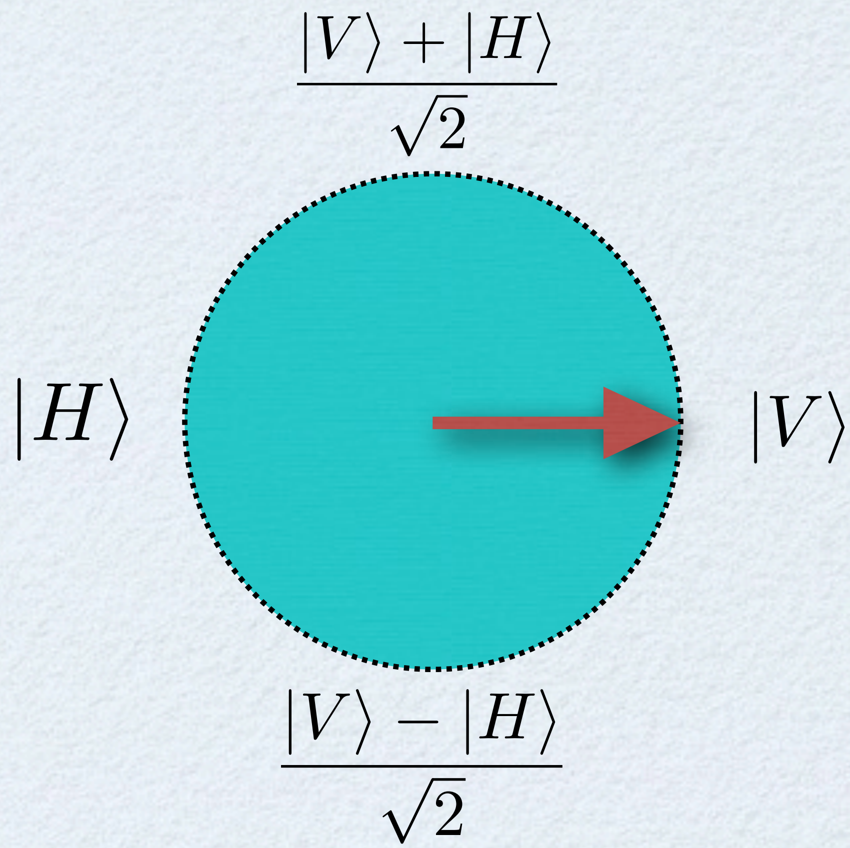
QUANTUM METROLOGY BOUNDS ON THE REPLICATION CAPACITY

Replicating clocks

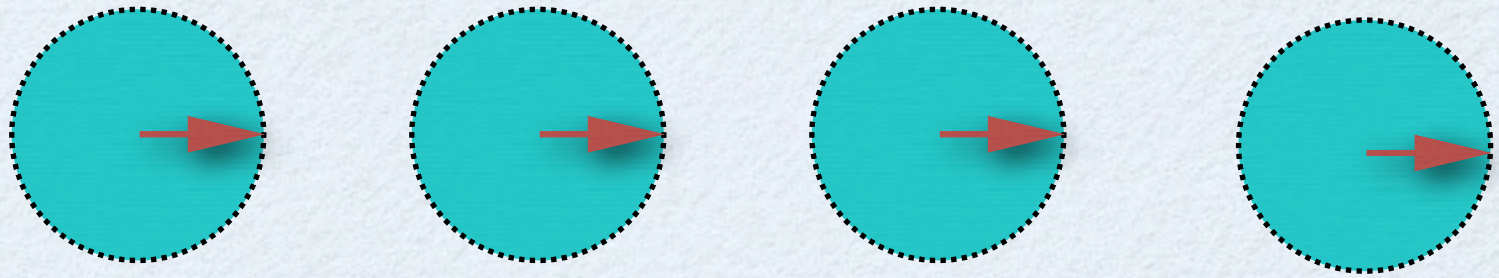
Clock states: $|\psi_t\rangle = e^{-itH}\,|\psi\rangle \qquad t \in \mathbb{R},\ H^\dagger = H$

Example: linearly polarized photons/preceding nuclear magnets

$$|\psi_t\rangle = \cos(\omega\, t)\,|0\rangle + \sin(\omega\, t)\,|1\rangle$$



Standard quantum limit

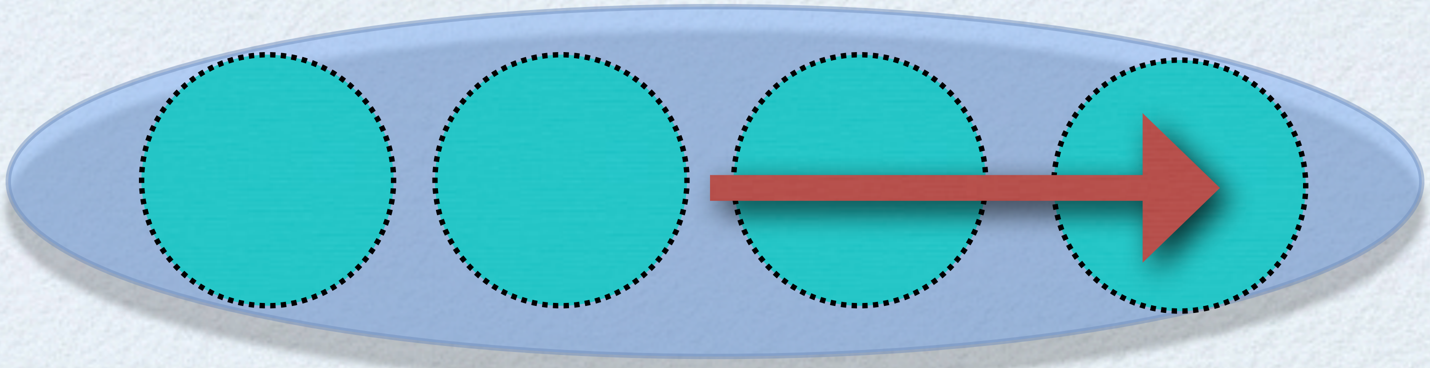


$$|\psi_t\rangle^{\otimes N}$$

Standard quantum limit (SQL):
For independently prepared clocks,
the error scales like the inverse of the number of clocks:

$$\mathrm{Var}(t) \approx \frac{\mathrm{const}}{N}$$

Heisenberg limit



$$\left(e^{-itH} \otimes \cdots \otimes e^{-itH}\right) |\Psi\rangle$$

With a suitable entangled state the variance scales as

$$\mathrm{Var}(t) \approx \frac{\mathrm{const}}{N^2}$$

Heisenberg limit: $1/N^2$ is the best scaling allowed by quantum mechanics

Ultimate limits on the replication capacity

Theorem (Chiribella-Yang-Yao, Nature Commun. 2013):
In finite dim, if a set of states contains a subset of clock states, then the replication capacity is upper bounded as

$$\alpha^* \leq 1 \quad \text{for deterministic processes}$$

and $$\alpha^* \leq 2 \quad \text{for probabilistic processes}$$

Corollary: deterministic processes can only produce a negligible amount of replicas.
No physical process can reliably replicate quantum information at a rate faster than quadratic.

ACHIEVING THE HEISENBERG LIMIT

Quadratic speedup in replication

For quantum clock states

there exists a probabilistic copy machine that obtains fidelity

$$F_{N \to M} = 1 - O\left[\exp\left(-\text{constant} \times \frac{N^2}{M}\right)\right]$$

The fidelity tends to 1 whenever M is of order smaller than N^2

faster than any inverse polynomial.

Super-replication

For clock states, one can probabilistically embezzle from Nature a number of extra-copies that is large compared to N

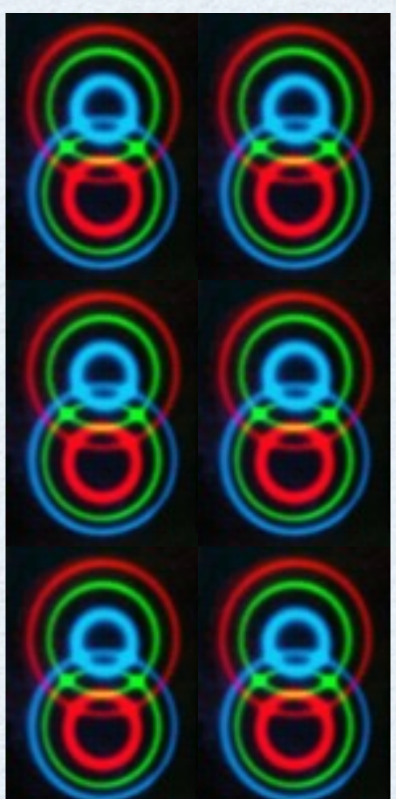
Super-replication: replication at rate $\alpha \geq 1$

Example: a probabilistic process can transform N = 100 linearly polarized photons into 1000 copies with fidelity F= 99.9%
The best deterministic process can only achieve F = 57%

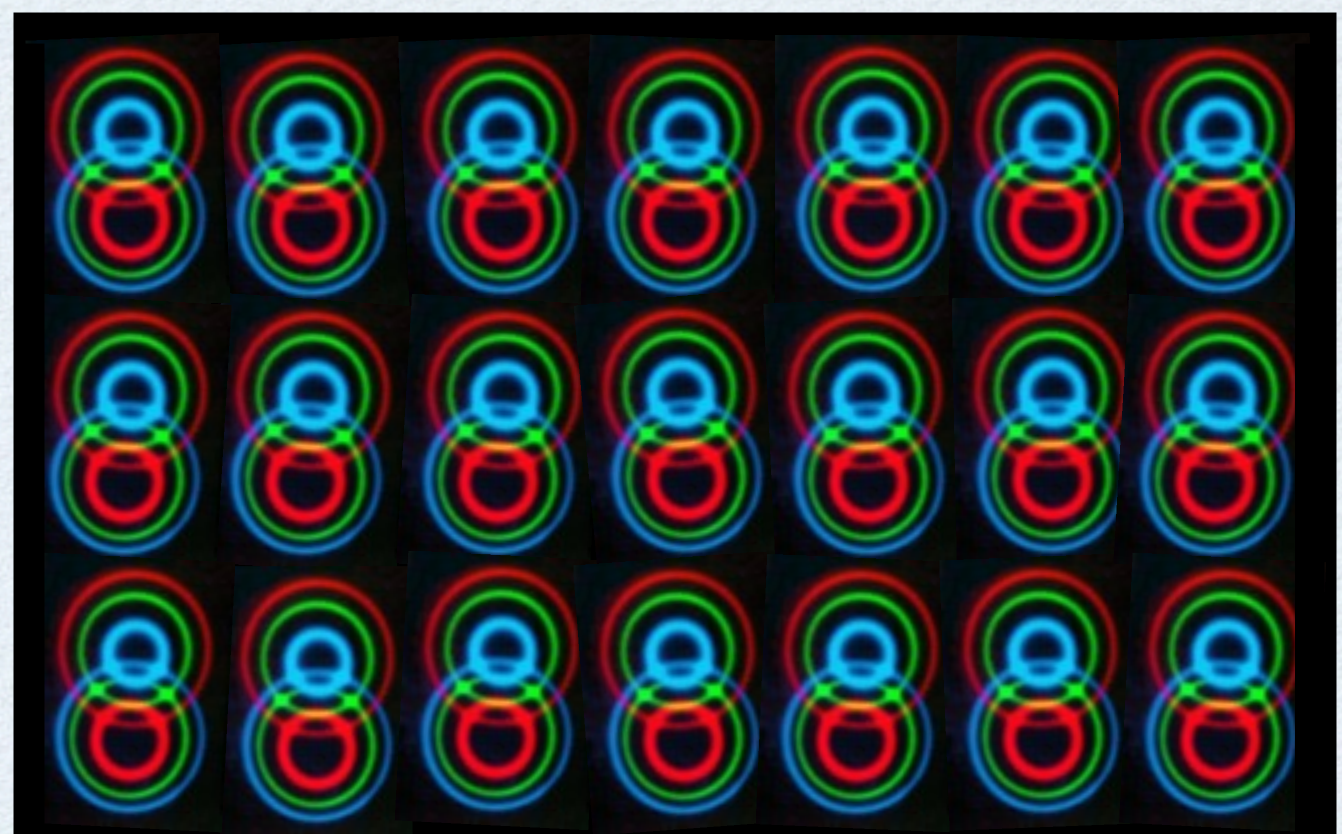
Replicating entanglement

Arbitrary maximally entangled states can be super-replicated probabilistically $\alpha^* = 2$

$$|\Phi_U\rangle := (U \otimes I)\,|\Phi\rangle$$



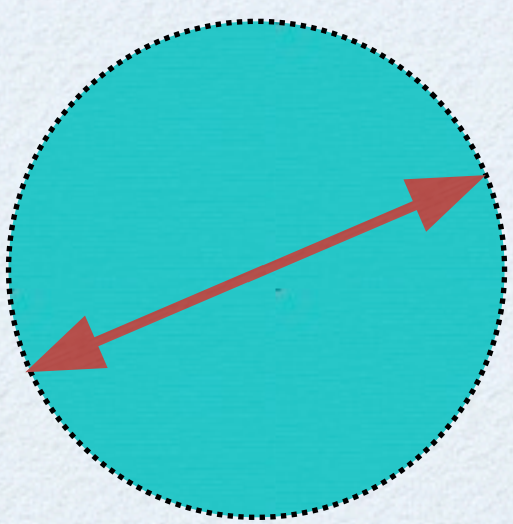
$\longrightarrow$



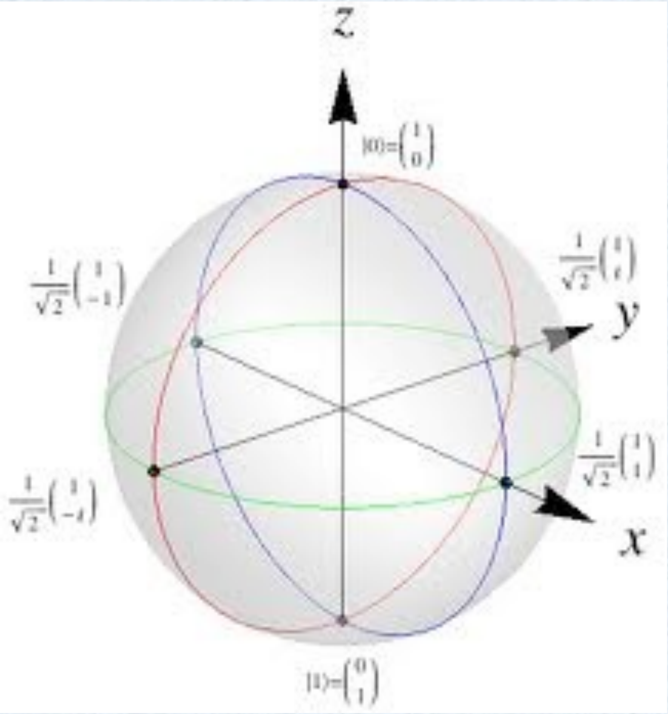
Universal super-replication?

Can we find a universal super-replicator?
a process that super-replicates arbitrary pure states?

No!   No advantage in using probabilistic processes for the replication of arbitrary pure states.
Replication of arbitrary pure states obeys the
standard quantum limit (replication rate <1)



circle:
super-replication
is possible



sphere:
super-replication
is impossible

RATE VS PROBABILITY: THE TRADEOFF

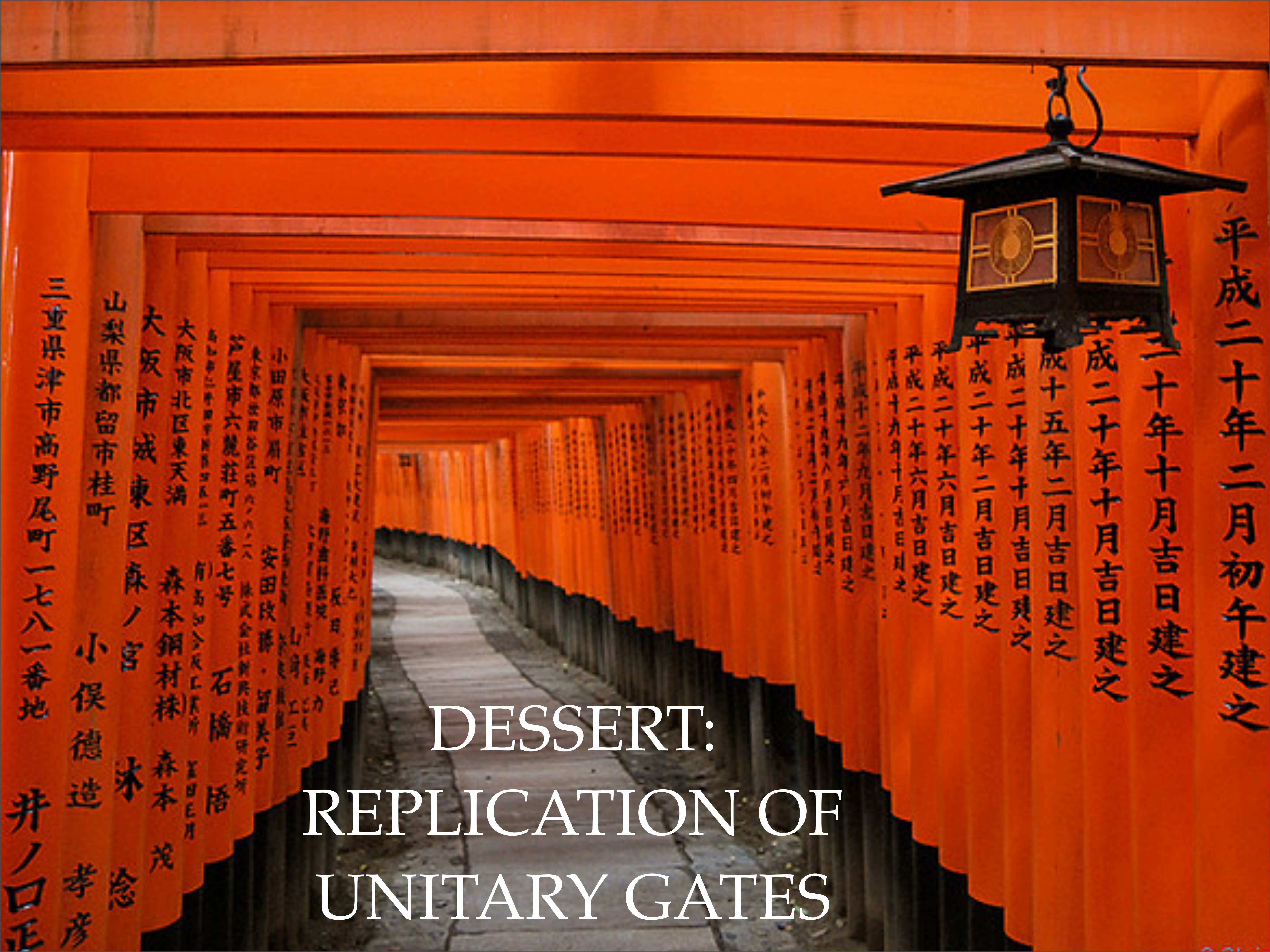
Strong converse of the metrology bounds

Unfortunately, nothing comes for free...

Super-replication comes with a curse:
small probability of success.

How small?

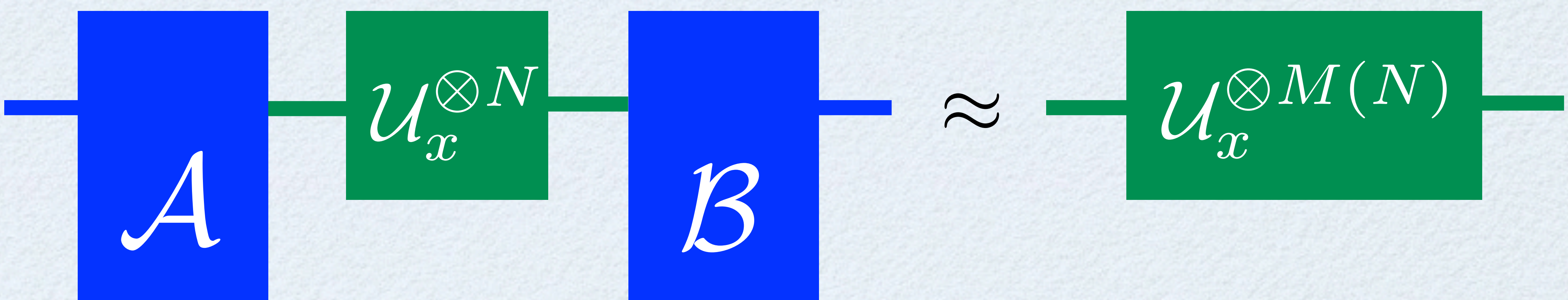
Theorem (Chiribella, Yang, Yao 2013)
For rate $\alpha = 1 + \epsilon$, $\epsilon < 1$
the success probability must vanish compared to $\exp[-N^{\epsilon}]$
Otherwise, the fidelity will go to zero.
For $\epsilon > 1$ the fidelity goes to zero (no matter what)



DESSERT: REPLICATION OF UNITARY GATES

Replicating unitary gates

Problem: build up a quantum network that simulates M uses of a gate by using it only N times.



$$\forall x \in \mathsf{X}$$

Replicating phase shift gates

Dür, Sekataski, Skotiniotis (PRL 2015)

Deterministic super-replication of phase shift gates

For gates of the form $U_t = e^{-itH}$

a deterministic network can obtain replication capacity

$$\alpha^* = 2$$

Is it possible to replicate arbitrary gates?

Universal super-replication

Theorem (Chiribella, Yang, Huang, PRL 2015)
For every finite-dimensional quantum system there exists a deterministic, universal network that achieves replication capacity $\alpha^* = 2$ for arbitrary unitary gates.
The quadratic replication rate is optimal:
every quantum network replicating gates at rate > 2 will necessarily have zero fidelity on most inputs.

No contradiction with the no-cloning theorem:
the gate U cannot be extracted deterministically from the state $|\psi_U\rangle := U\,|\psi\rangle$

Super-generation of maximally entangled states

Using an unknown unitary gate U for N times one can generate deterministically M>N approximate copies of the maximally entangled state

$$|\Phi_U\rangle := (U \otimes I)\,|\Phi\rangle$$

with fidelity

$$F^{\mathrm{ent}}_{\mathrm{gen}}[N \to M] \geq 1 - 2(M+1)\,\exp\left[-\frac{N^2}{2M}\right]$$

CONCLUSIONS

Conclusions

- Fundamental limits to probabilistic amplification:
  -optimality of RL08 for Gaussian distributed CS
  -probabilistic benchmarks

- Ultimate limits to the replication rates set by the limits of quantum metrology:
  SQL: rate <1 for deterministic processes (negligible extra-copies)
  HL: rate <2 for probabilistic processes (quadratic speed-up)

- Breaking the SQL: Super-replication
  embezzling from Nature a large number of nearly perfect copies.
  Can be done for clock states and max ent states.

- Universal gate replication: simulates up to a quadratic number of gate uses without violating the no-cloning theorem.

THANK YOU FOR YOUR ATTENTION!

千人計劃

RECRUITMENT PROGRAM OF GLOBAL EXPERTS

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Proof idea (qubit case)

- think of qubits as spin 1/2 particles
- for a system of K qubits, let $P_J^{(K)}$ be the projector on the subspaces with quantum number of the angular momentum less than J
- define the encoding channel

$$\mathcal{E}_J^{(K)}(\rho) := P_J^{(K)} \rho P_J^{(K)} + \mathrm{Tr}\left[\left(I^{\otimes K} - P_J^{(K)}\right)\rho\right] \rho_0$$

- consider the fidelity between a generic state $|\Psi\rangle$ and its encoded version $\mathcal{E}_J^{(K)}\left(|\Psi\rangle\langle\Psi|\right)$

Proof idea, cont'd

- Theorem

If $|\Psi\rangle$ is chosen uniformly at random, then for every $\epsilon > 0$ one has

$$\mathsf{Prob}\left[F_{\Psi}^{(K,J)} < 1 - \epsilon\right] < \frac{2(K+1)}{\epsilon} \exp\left[-\frac{2J^2}{K}\right]$$

a random state of M qubits can be encoded with little error into a subspace with angular momentum at most $\sqrt{M}$,
wherein the action of the M gates can be simulated via the action of $\sqrt{M}$ gates