# Hierarchy of efficiently computable and faithful lower bounds to quantum discord

## arXiv:1501.06855

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operational
significance?
(almost) everywhere
= worthless?

If there are ubiquitous
– and without doubt –
quantum features,
I do want to understand them!

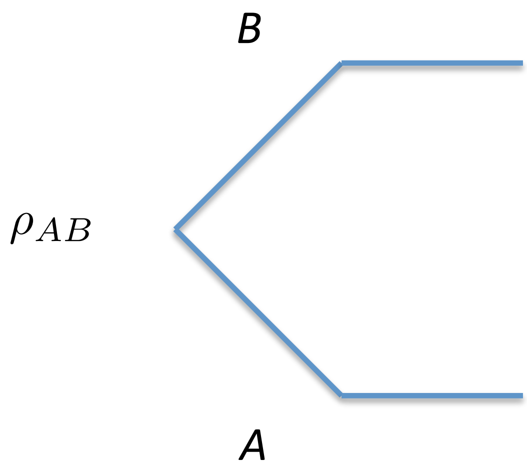
- Focus on basic operational tasks / properties that distinguish the quantum from the classical

- Be quantitative – not just qualitative

- Connect with recent QIP results and modern techniques (like semidefinite programming)

Focus on basic operational task
that distinguishes
the quantum from the classical

# On what it means for correlations to be(have) classical(ly)



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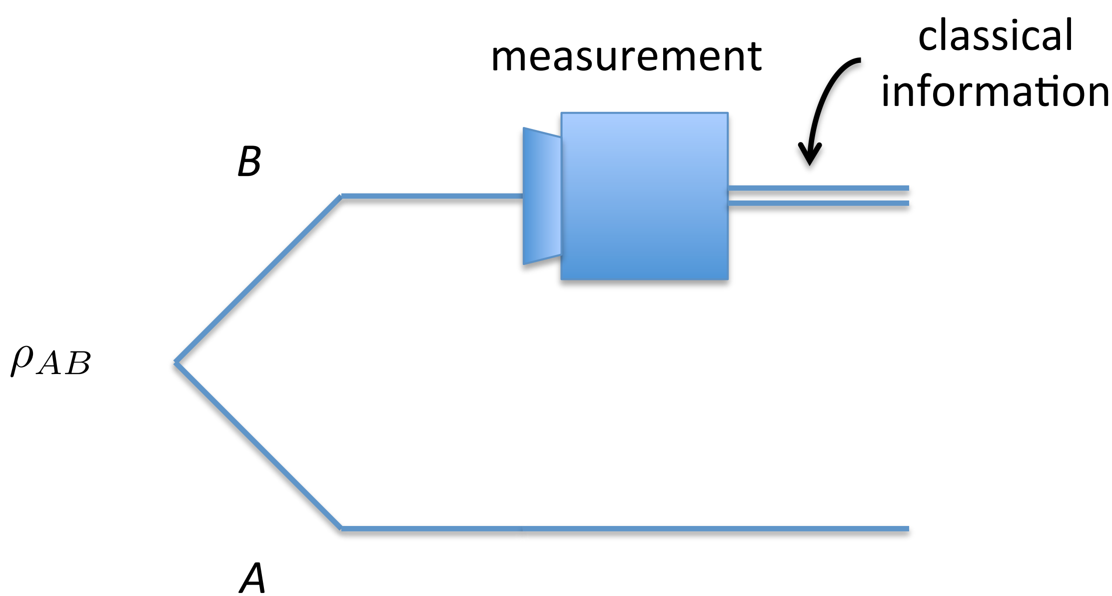
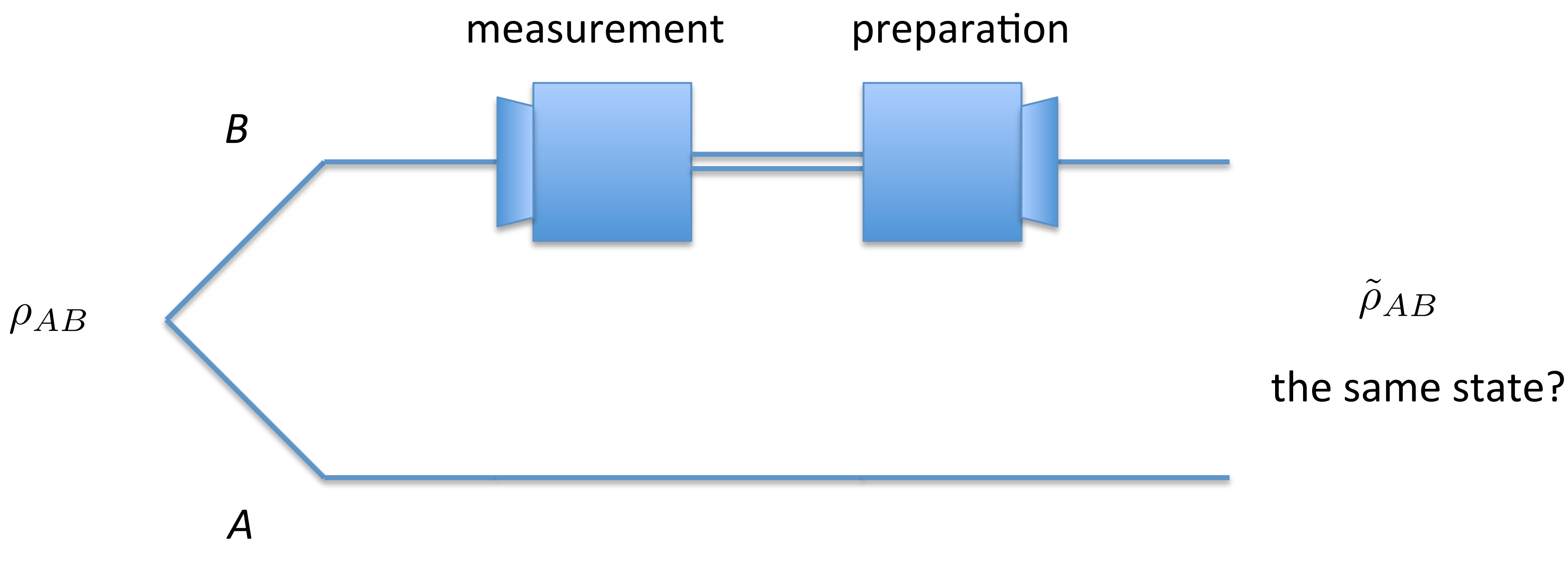
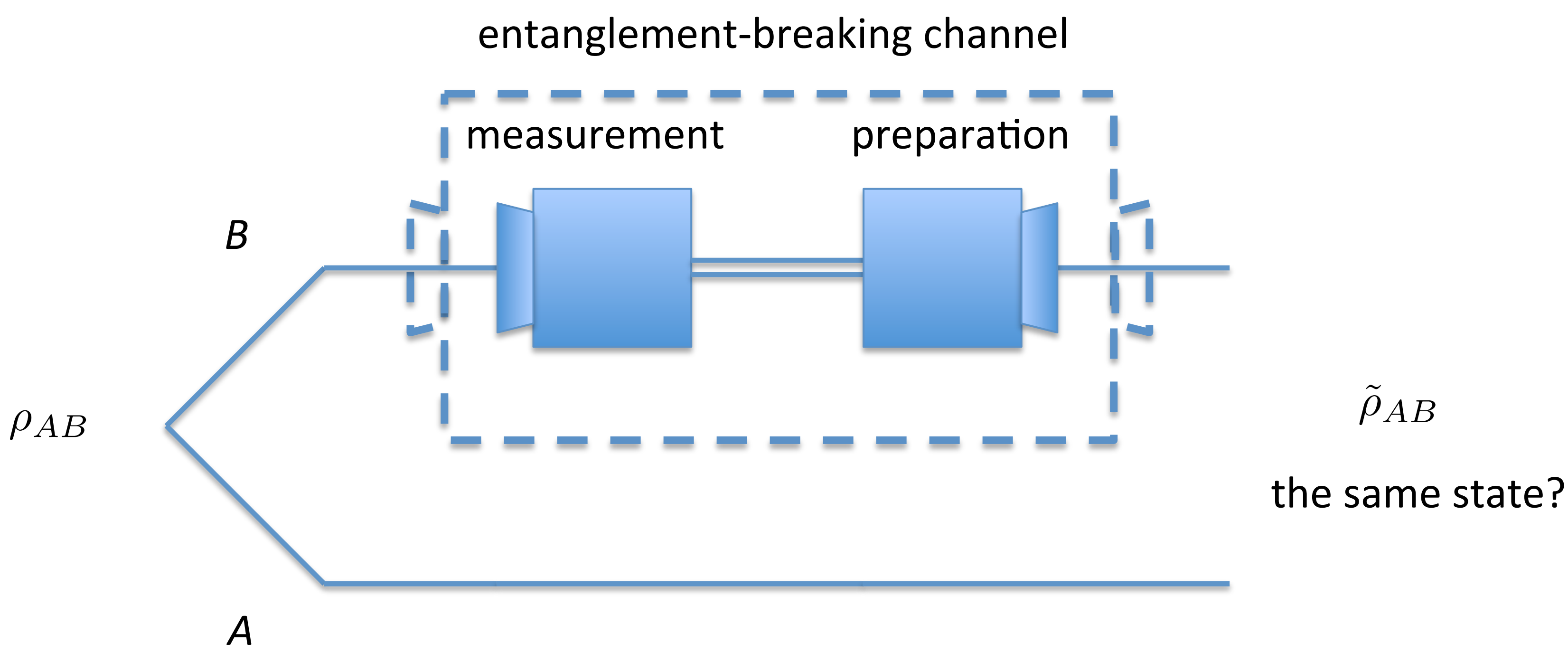



the same amount of correlations?

# On what it means for correlations to be(have) classical(ly)



# On what it means for correlations to be(have) classical(ly)



If the quantification of correlations is in terms of **mutual information**,

$$I(A : B) = S(A) + S(B) - S(AB)$$

with the von Neumann entropy

$$S(X) = S(\rho_X) = -\operatorname{Tr}(\rho_X \log \rho_X)$$

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$$S(X) = S(\rho_X) = -\operatorname{Tr}(\rho_X \log \rho_X)$$

*the answer ( "yes" / "no" ) is the same from either perspective!*

(We know this thanks to a powerful result by Petz)

The following are equivalent:

There exists an entanglement breaking channel $\Lambda^{\mathrm{EB}}$

such that $\Lambda_B^{\mathrm{EB}}[\rho_{AB}] = \rho_{AB}$

There exists a measurement $\mathcal{M}_{B \to Y}$

such that $I(A:Y)_{\mathcal{M}_{B \to Y}[\rho_{AB}]} = I(A:B)_{\rho_{AB}}$

*B* is classical to begin with, i.e.

$$\rho_{AB} = \sum_b p_b \rho_A^b \otimes |b\rangle\langle b|_B$$

# Quantum discord

$$D(A:\underline{B})_\rho := \min_{\mathcal{M}_{B\to Y}} \big( I(A:B)_{\rho_{AB}} - I(A:Y)_{\mathcal{M}_{B\to Y}[\rho_{AB}]} \big)$$

[Olliver&Zurek, PRL 2001; Henderson&Vedral, JPA 2001]

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The following are equivalent:

There exists an entanglement breaking channel $\Lambda^{\mathrm{EB}}$

such that $\Lambda_B^{\mathrm{EB}}[\rho_{AB}] = \rho_{AB}$

$\rho_{AB}$ has vanishing discord on *B*

*B* is classical to begin with, i.e.

$$\rho_{AB} = \sum_b p_b \rho_A^b \otimes |b\rangle\langle b|_B$$

The following are equivalent:

It is possible to store / transmit classically the *B* part of $\rho_{AB}$

!

$\rho_{AB}$ has vanishing discord on *B*

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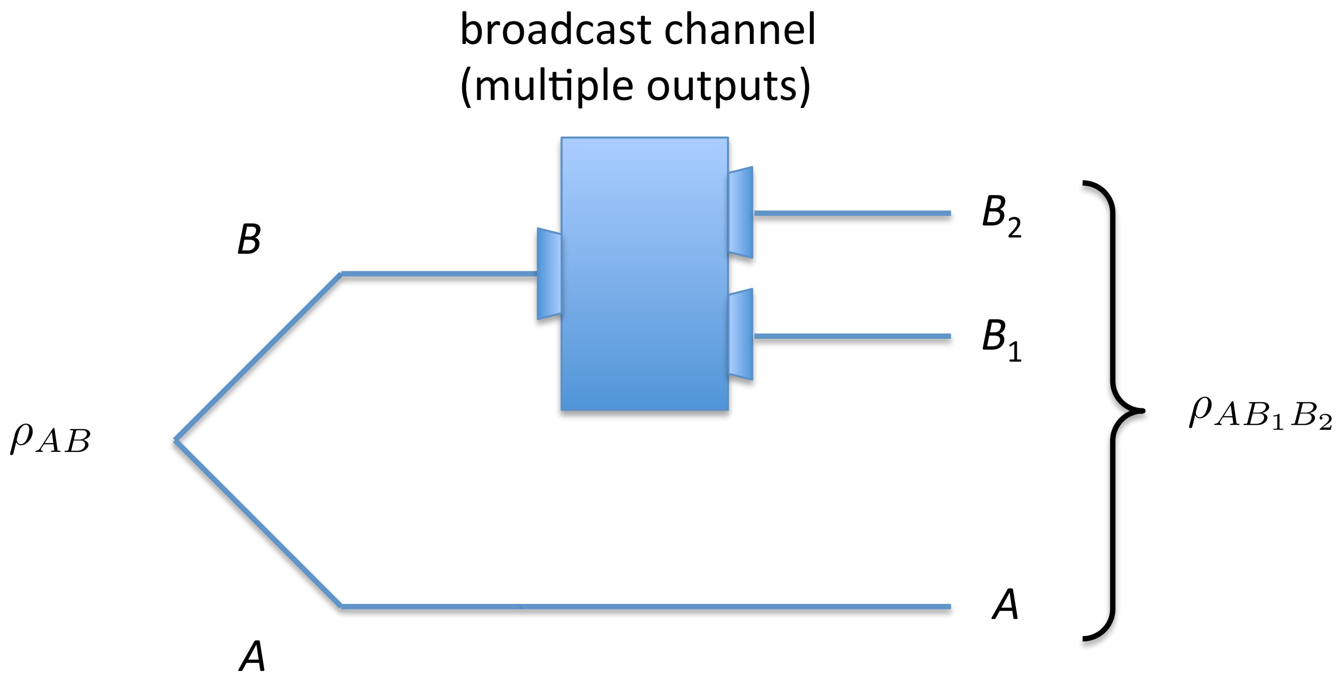
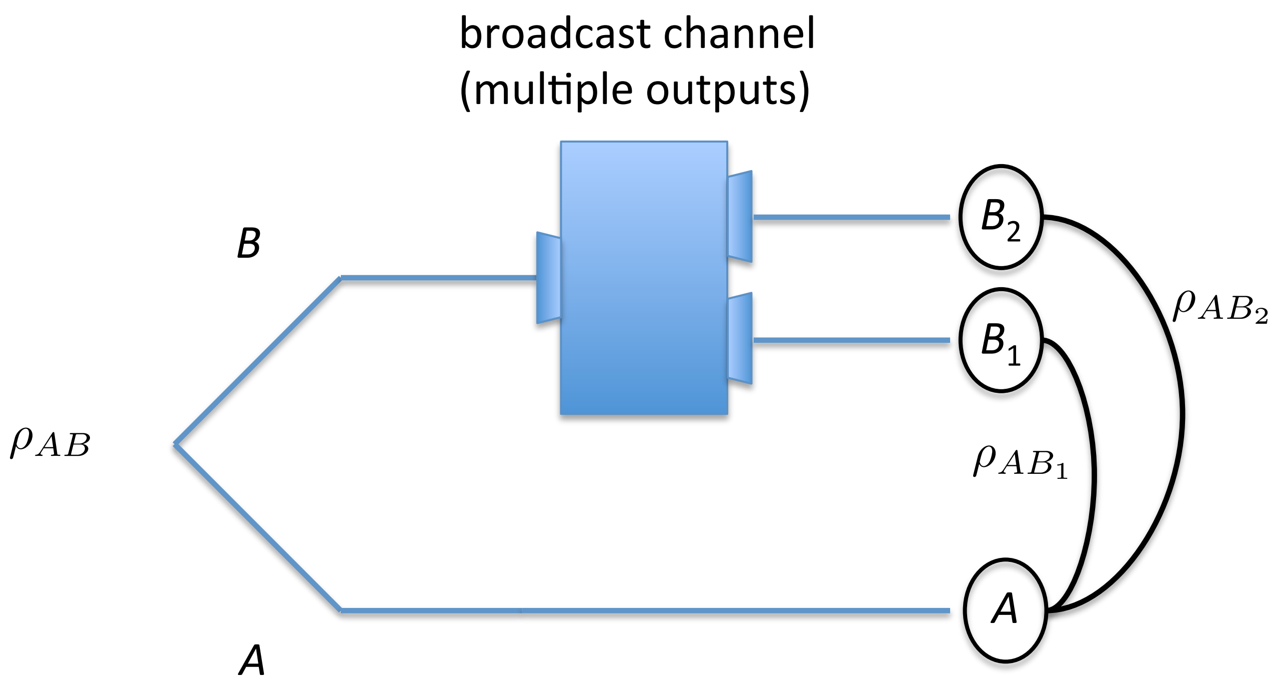
# On what it means for correlations to be(have) classical(ly)



# On what it means for correlations to be(have) classical(ly)




Are $\rho_{AB_1}$ and $\rho_{AB_2}$ equal to $\rho_{AB}$ ?

Do they contain the same amount of correlations as $\rho_{AB}$ ?

The following are equivalent:

It is possible to store / transmit classically the *B* part of $\rho_{AB}$

!

$\rho_{AB}$ has vanishing discord on *B*

*B* is classical to begin with, i.e.

$$\rho_{AB} = \sum_b p_b \rho_A^b \otimes |b\rangle\langle b|_B$$

The following are equivalent:

It is possible to store / transmit classically the *B* part of $\rho_{AB}$ !

$\rho_{AB}$ has vanishing discord on *B*

*B* is classical to begin with, i.e.

$$\rho_{AB} = \sum_b p_b \rho_A^b \otimes |b\rangle\langle b|_B$$

$\rho_{AB}$ (equivalently, its correlations) can be locally broadcast on side *B* !

Be quantitative rather than qualitative

# Mutual information and perfect recoverability

Conditional mutual information

$$I(A:B|C) := I(A:BC) - I(A:C)$$

Strong subadditivity (a.k.a. "losing pieces does not increase correlations")

$$I(A:B|C) \geq 0$$

Corollary of Petz theorem:

$I(A:B|C) = 0$ $\Longleftrightarrow$ there is a recovery channel $\mathcal{R}_{C\to BC}$ such that $\mathcal{R}_{C\to BC}[\rho_{AC}] = \rho_{ABC}$

# Mutual information and approximate recoverability

There is a recovery channel $\mathcal{R}_{C\to BC}$ such that

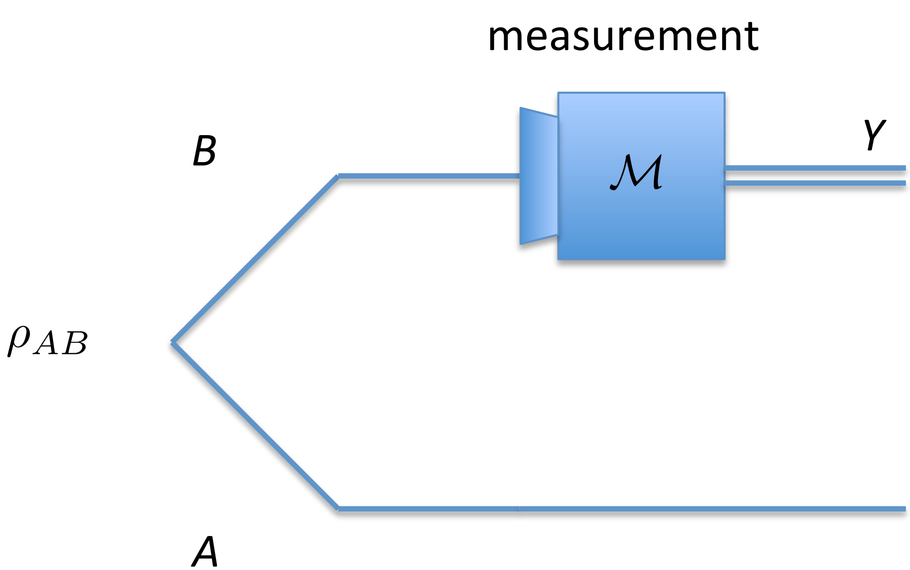
$$F(\mathcal{R}_{C\to BC}[\rho_{AC}], \rho_{ABC}) \geq 2^{-\frac{1}{2}I(A:B|C)_{\rho_{ABC}}}$$

with the fidelity $F(\sigma,\rho) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$

**Fawzi-Renner 2014**

# Discord as loss of correlations

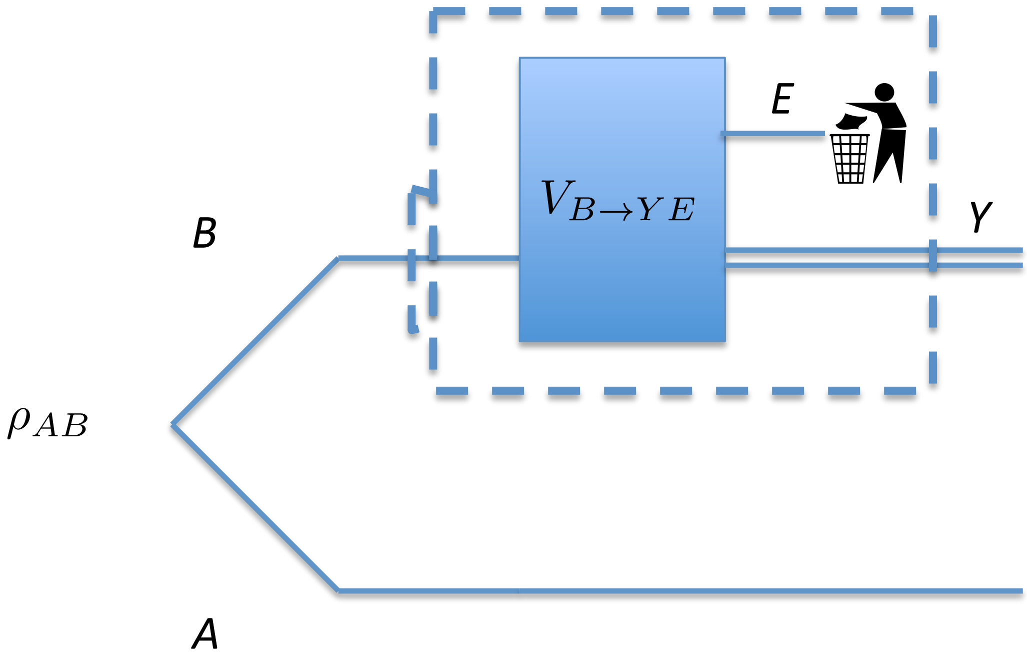
$$D(A:\underline{B})_\rho := \min_{\mathcal{M}_{B\to Y}} \big( I(A:B)_{\rho_{AB}} - I(A:Y)_{\mathcal{M}_{B\to Y}[\rho_{AB}]} \big)$$



# Discord as loss of correlations

$$D(A:\underline{B})_\rho = \min_{V_{B\to YE}} I(A:E|Y)_{\rho_{AYE}}$$

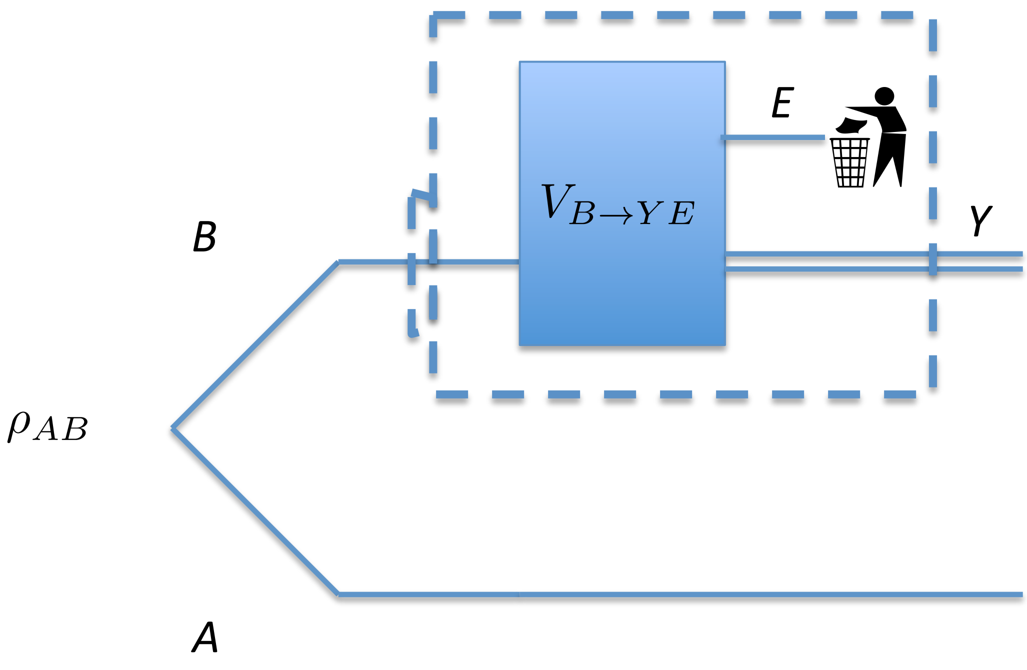
$$\mathcal{M}_{B\to Y}[\cdot] = \mathrm{Tr}_E(V_{B\to YE}\cdot V^\dagger_{B\to YE})$$



# Discord as loss of correlations

loss
=
measurement

recovery
=
preparation



# Discord as loss of correlations

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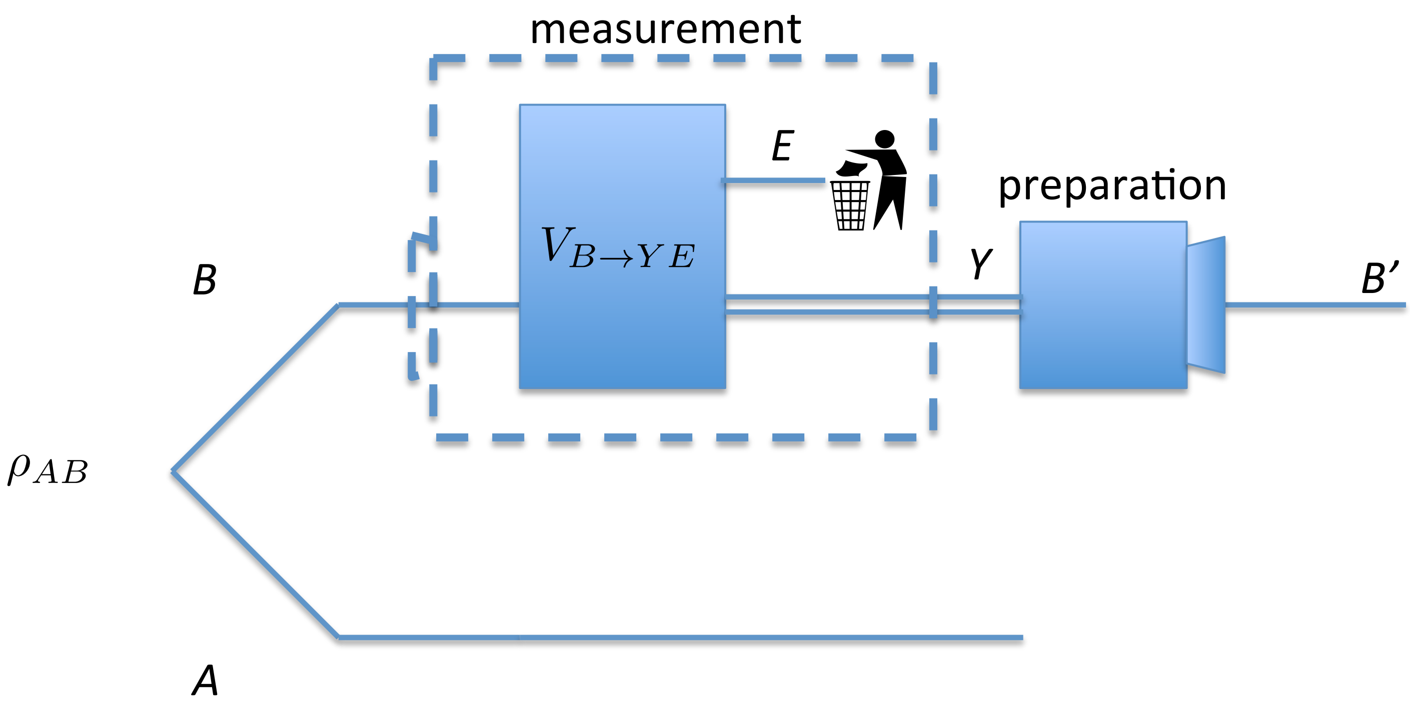
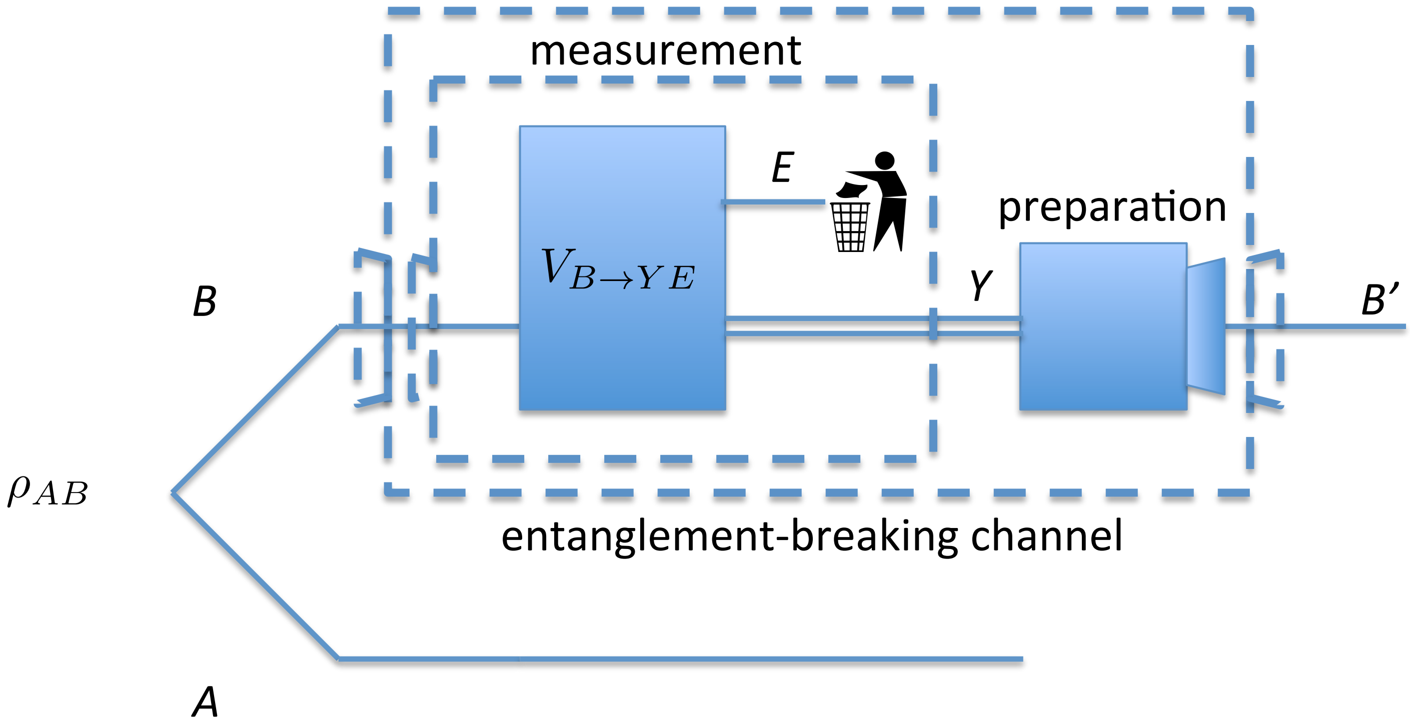


# Discord as loss of correlations (II)

loss
=
measurement

recovery
=
preparation



# Surprisal of measurement recoverability

Using Fawzi-Renner [Seshadreesan and Wilde, 2014]

$$\sup_{\Lambda^{\mathrm{EB}} \in \mathcal{L}^{\mathrm{EB}}} F(\Lambda_B^{\mathrm{EB}}[\rho_{AB}], \rho_{AB}) \geq 2^{-\frac{1}{2} D(A:\underline{B})}$$

entanglement breaking channel

class of entanglement breaking channels

# Surprisal of measurement recoverability

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$$\sup_{\Lambda^{\mathrm{EB}} \in \mathcal{L}^{\mathrm{EB}}} F(\Lambda_B^{\mathrm{EB}}[\rho_{AB}], \rho_{AB}) \geq 2^{-\frac{1}{2} D(A:\underline{B})}$$

Small discord implies the possibility of transmitting / storing part of a quantum system classically with high fidelity

*!*

(Converse also true: high discord → low fidelity)

# Surprisal of measurement recoverability

Using Fawzi-Renner [Seshadreesan and Wilde, 2014]

$$\sup_{\Lambda^{\mathrm{EB}} \in \mathcal{L}^{\mathrm{EB}}} F(\Lambda_B^{\mathrm{EB}}[\rho_{AB}], \rho_{AB}) \geq 2^{-\frac{1}{2} D(A:\underline{B})}$$

Introducing the **suprisal of measurement recoverability** [Seshadreesan and Wilde, 2014]

$$D_F(A:\underline{B}) := -\log \sup_{\Lambda^{\mathrm{EB}} \in \mathcal{L}^{\mathrm{EB}}} F^2(\Lambda_B^{\mathrm{EB}}[\rho_{AB}], \rho_{AB})$$

one can write

$$D_F(A:\underline{B}) \leq D(A:\underline{B})$$

# Remarks

- The surprisal of measurement recoverability is a good "quantumness of correlations" / discord-like measure in itself
  It actually has an operational meaning that we (can) use to motivate discord!

- Neither the discord proper nor the surprisal of measurement recoverability are easily computed (either analytically or numerically)
  General lower bounds were also lacking

- Many other discord-like measures have been proposed, but:
  - not necessarily related to fundamental quantum information processing tasks / fundamental quantum properties
  - not necessarily easy/easier to compute in general cases

Semidefinite-programming techniques to study discord

# A general paradigm

$$D_{F,\mathcal{L}}(A:\underline{B}) := -\log \sup_{\Lambda\in\mathcal{L}} F^2(\Lambda_B[\rho_{AB}], \rho_{AB})$$

channel

interesting class of channels

( Can be further generalized! [P, Narasimhachar, Calsamiglia, NJP 2014] )

# A general paradigm

Surprisal of measurement recoverability is special case

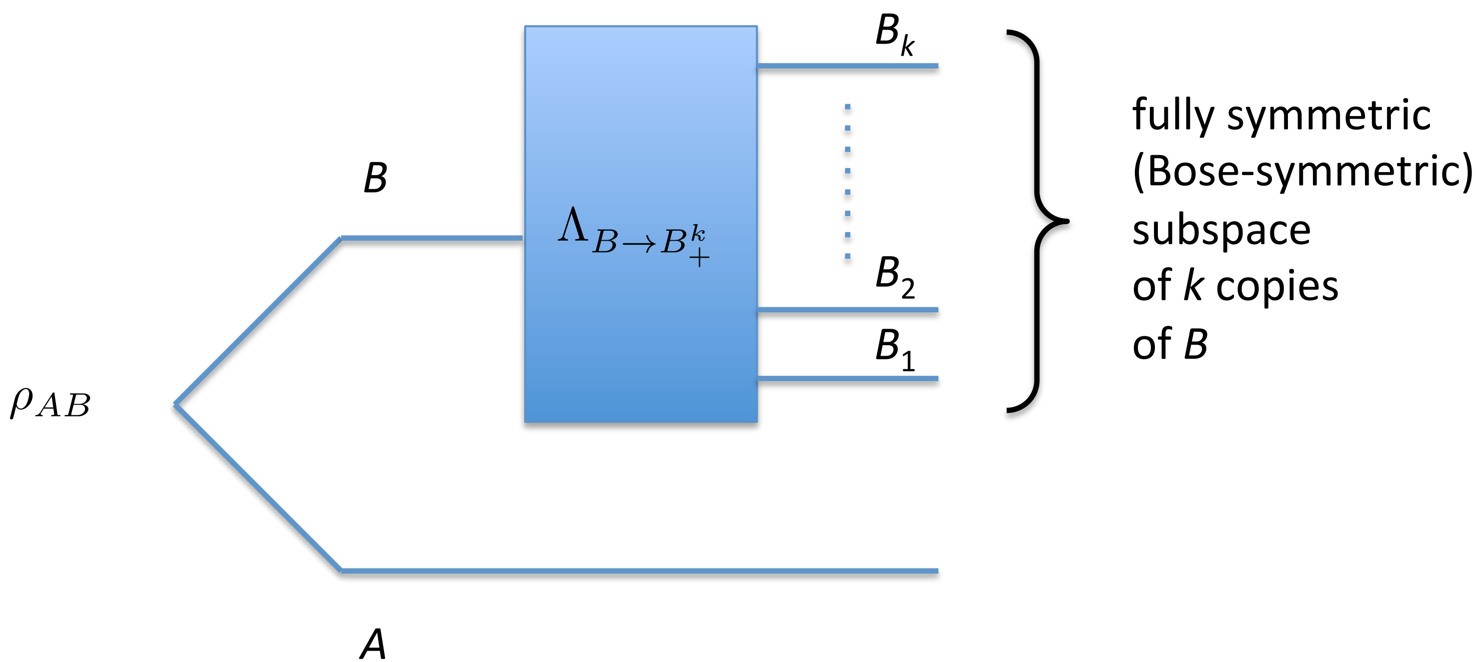
$$D_F(A:\underline{B}) \mapsto D_{F,\mathcal{L}^{\mathrm{EB}}}(A:\underline{B})$$

class of entanglement
breaking channels

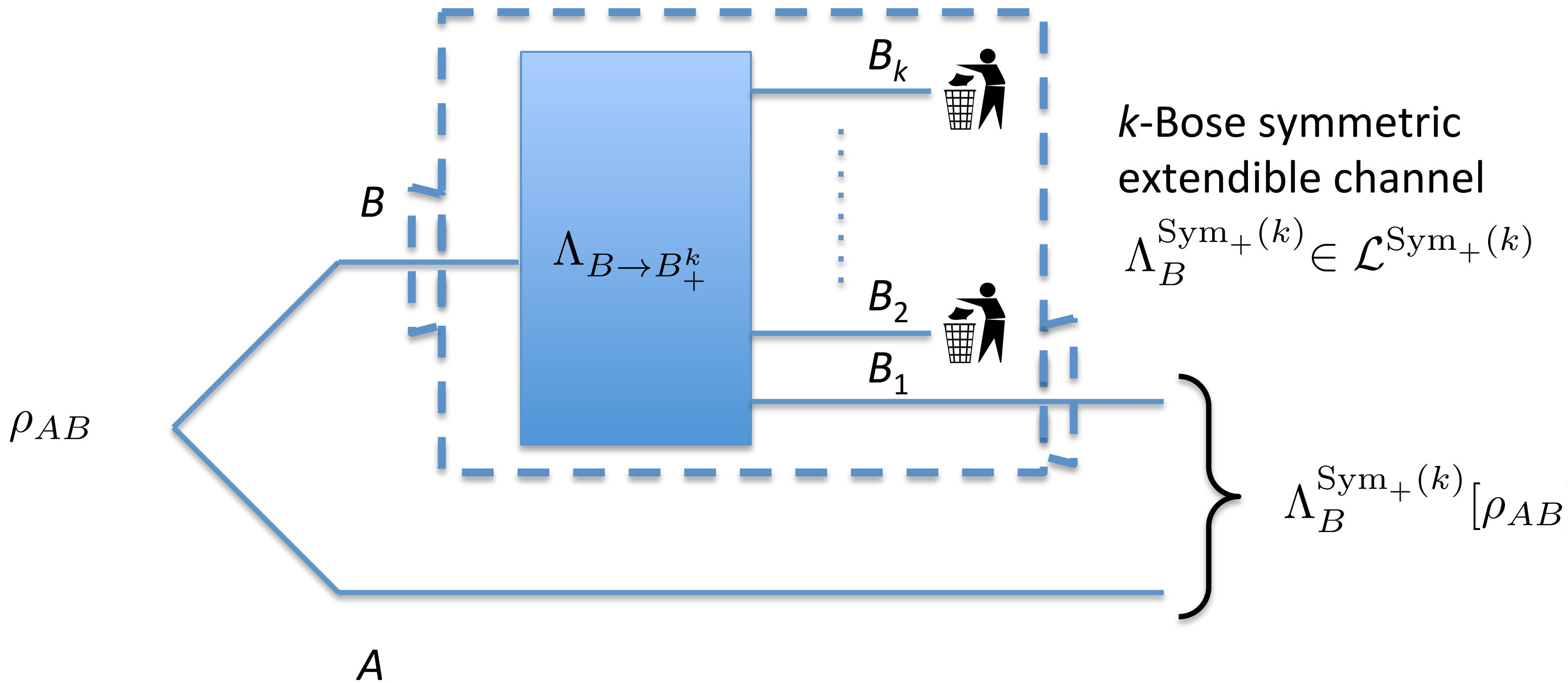
Furthermore

$$\mathcal{L}^{\mathrm{EB}} \subseteq \mathcal{L} \quad \Longrightarrow \quad D_{F,\mathcal{L}}(A:\underline{B}) \leq D_{F,\mathcal{L}^{\mathrm{EB}}}(A:\underline{B}) \leq D(A:\underline{B}).$$

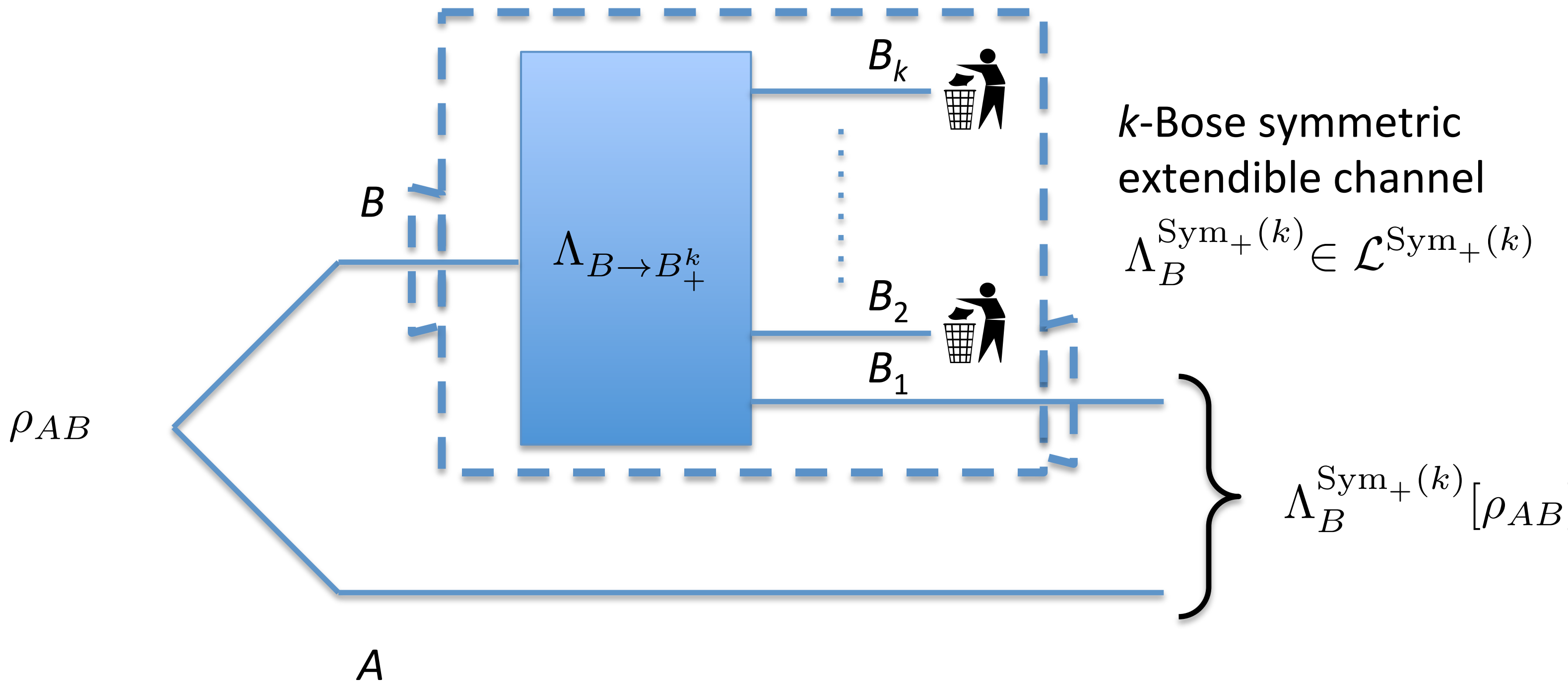
# Symmetric broadcasting



# Symmetric broadcasting



# Symmetric broadcasting




Every entanglement-breaking channel is ∞-Bose symmetric extendible

$$\mathcal{L}^{\mathrm{EB}} \subset \mathcal{L}^{\mathrm{Sym}_+(k)}$$

# Symmetric broadcasting

Furthermore there is an entanglement-breaking channel $\Lambda_B^{\mathrm{EB}}$ such that [Chiribella 2011]

$$\sup_{\rho_{AB}} \Delta\left(\Lambda_B^{\mathrm{Sym}_+(k)}[\rho_{AB}], \Lambda_B^{\mathrm{EB}}[\rho_{AB}]\right) \leq \frac{d_B}{k}$$

trace distance

$\mathcal{L}^{\mathrm{Sym}_+(k)} \to \mathcal{L}^{\mathrm{EB}}$ for $k \to \infty$

$D_{F,\mathcal{L}^{\mathrm{Sym}_+(k)}}(A:\underline{B}) \to D_{F,\mathcal{L}^{\mathrm{EB}}}(A:\underline{B})$ for $k \to \infty$

# Remarks

- It is a hierarchy

$$D_{F,\mathcal{L}^{\mathrm{Sym}_+(k)}}(A:\underline{B}) \leq D_{F,\mathcal{L}^{\mathrm{Sym}_+(k+1)}}(A:\underline{B})$$

- Every element of the hierarchy is a valid discord-like measure

- Even the lowest level of the hierarchy provide a faithful lower bound to discord (from no-local broadcasting)

$$D_{F,\mathcal{L}^{\mathrm{Sym}_+(2)}}(A:\underline{B}) = 0 \quad \Longleftrightarrow \quad D(A:\underline{B}) = 0$$

# An SDP program for fidelity

Fidelity $F(\sigma, \rho) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$

corresponds to the SDP

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}(\mathrm{Tr}(X) + \mathrm{Tr}(X^\dagger)) \\ \text{subject to} & \begin{pmatrix} \rho & X \\ X^\dagger & \sigma \end{pmatrix} \geq 0 \end{array}$$

[Watrous 2012, Killoran 2012]

# An SDP program for $D_{F,\mathcal{L}^{\mathrm{Sym}_+(k)}}(A : \underline{B})$

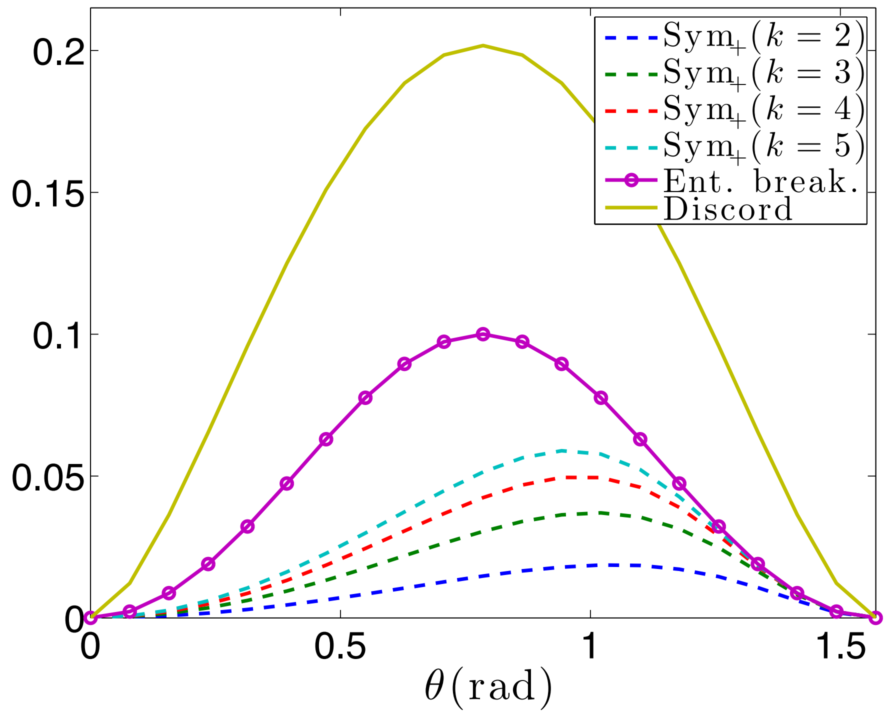
$$
\begin{aligned}
\text{maximize} \quad & \frac{1}{2}(\mathrm{Tr}(X) + \mathrm{Tr}(X^\dagger)) \\
\text{subject to} \quad & \begin{pmatrix} \rho_{AB} & X \\ X^\dagger & \mathrm{Tr}_{\setminus AB_1}(W_{BB^k}^{\Gamma_B} \rho_{AB}) \end{pmatrix} \geq 0 \\
& W_{BB^k} \geq 0 \\
& W_B = I_B \\
& W_{BB^k} = \Pi^+_{B^k} W_{BB^k} \Pi^+_{B^k}.
\end{aligned}
$$

computes fidelity between input state and output state

parameterizes the extendible channel via the Choi-Jamiolkowski isomorphism

$\Lambda_B^{\mathrm{Sym}_+(k)}[\rho_{AB}]$

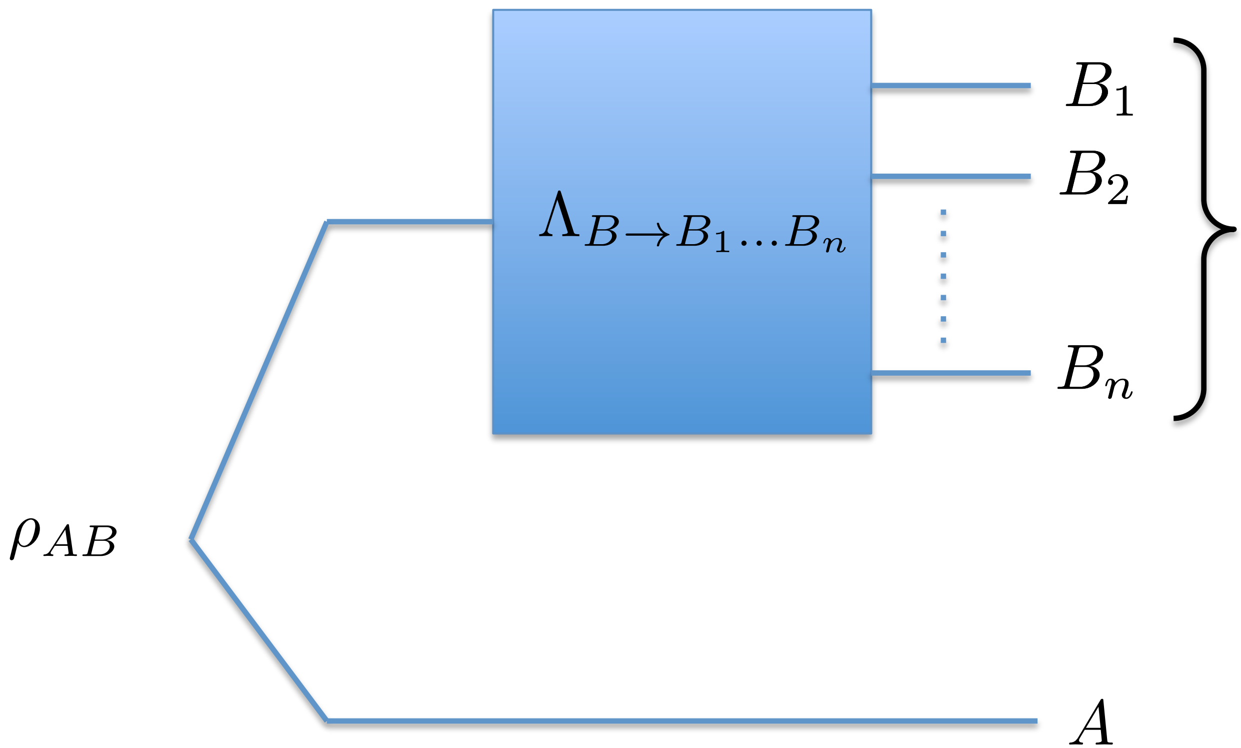
# An example




$$\rho_{AB}(\theta) = \frac{1}{2}|0\rangle\langle 0|_A \otimes |\psi_0(\theta)\rangle\langle\psi_0(\theta)|_B + \frac{1}{2}|1\rangle\langle 1|_A \otimes |\psi_1(\theta)\rangle\langle\psi_1(\theta)|_B$$

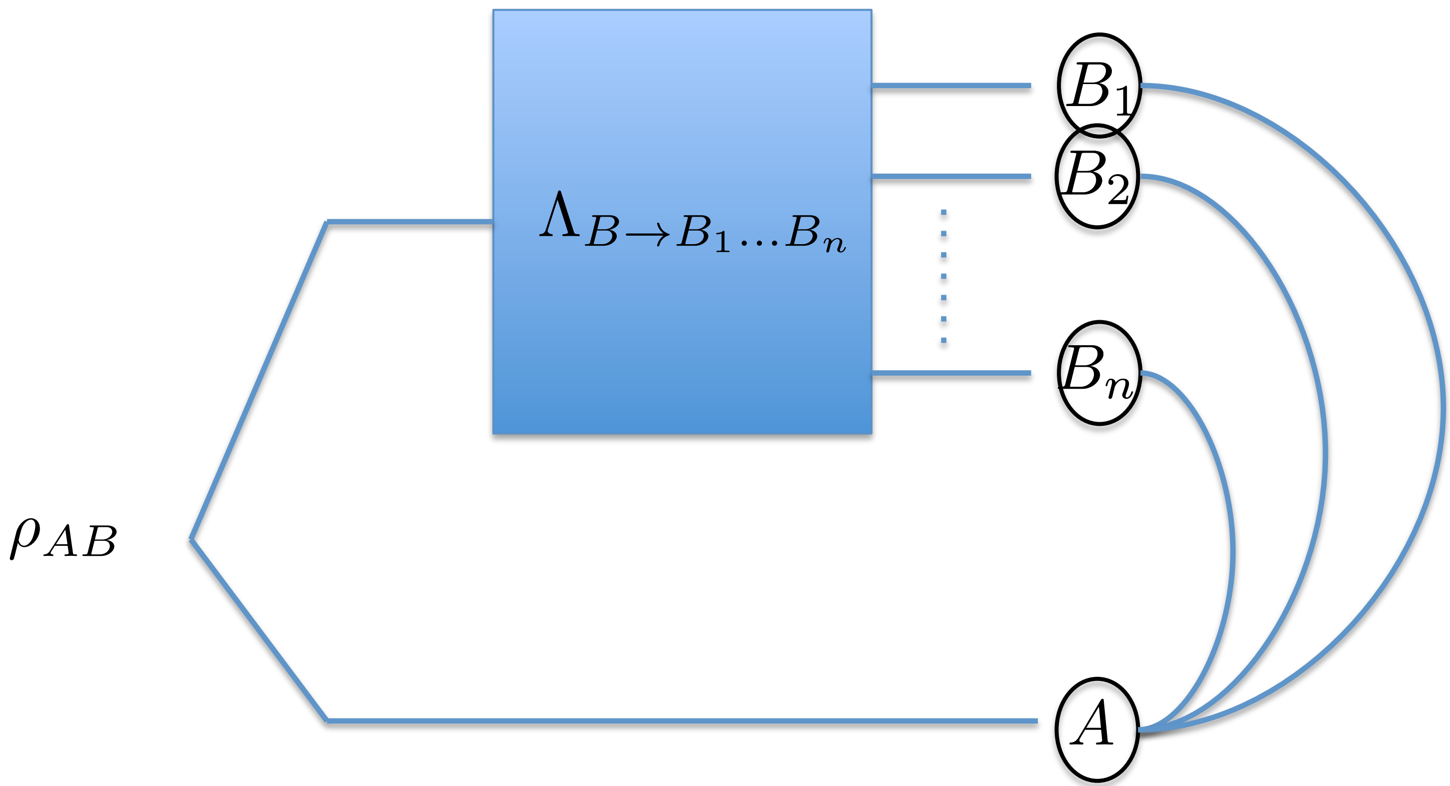
with $\quad |\psi_a(\theta)\rangle = \cos(\theta/2)|0\rangle + (-1)^a \sin(\theta/2)|1\rangle$

# General broadcasting of correlations




no constraint on the output (no symmetry, no bound on dimensions)

# General broadcasting of correlations

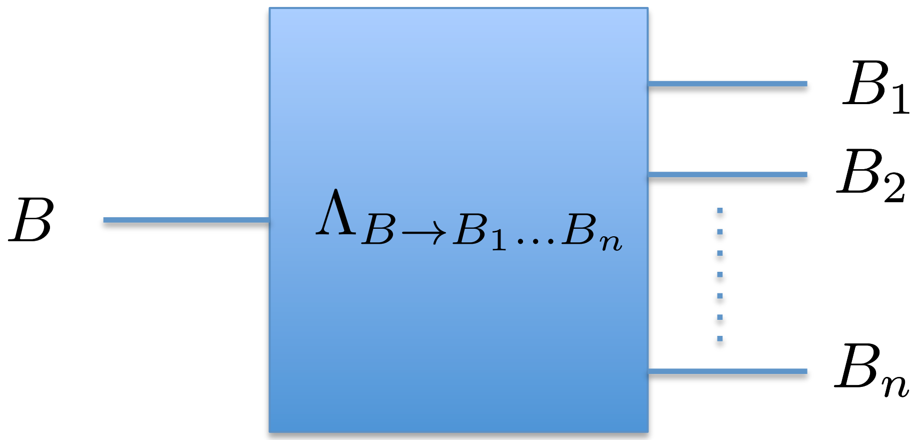



$$I(A : B_j) \leq I(A : B)$$

# An operational interpretation of quantum discord

$$\inf_{\Lambda_{A \to B_1 \ldots B_n}} \underbrace{\frac{1}{n} \sum_{j=1}^{n} (I(A:B) - I(A:B_j))}_{\text{average loss of correlations}} \xrightarrow[n \to \infty]{} \underbrace{I(A:B) - \max_{\mathcal{M}} I(A:B)_{\underbrace{(I_A \otimes \mathcal{M}_B)}_{\text{measurement}}[\rho_{AB}]}}_{\text{quantum discord}}$$

[Brandao, P, Horodecki, arXiv:1310.8640]

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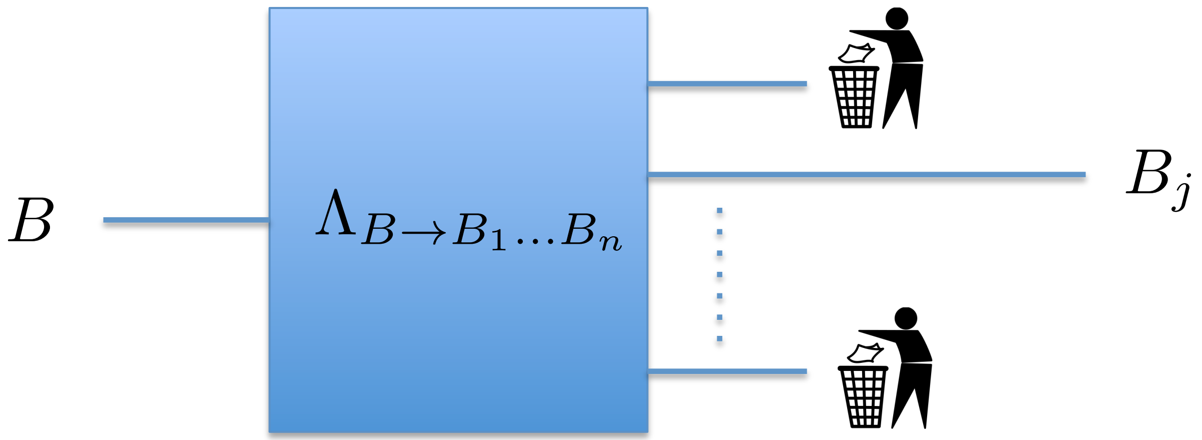


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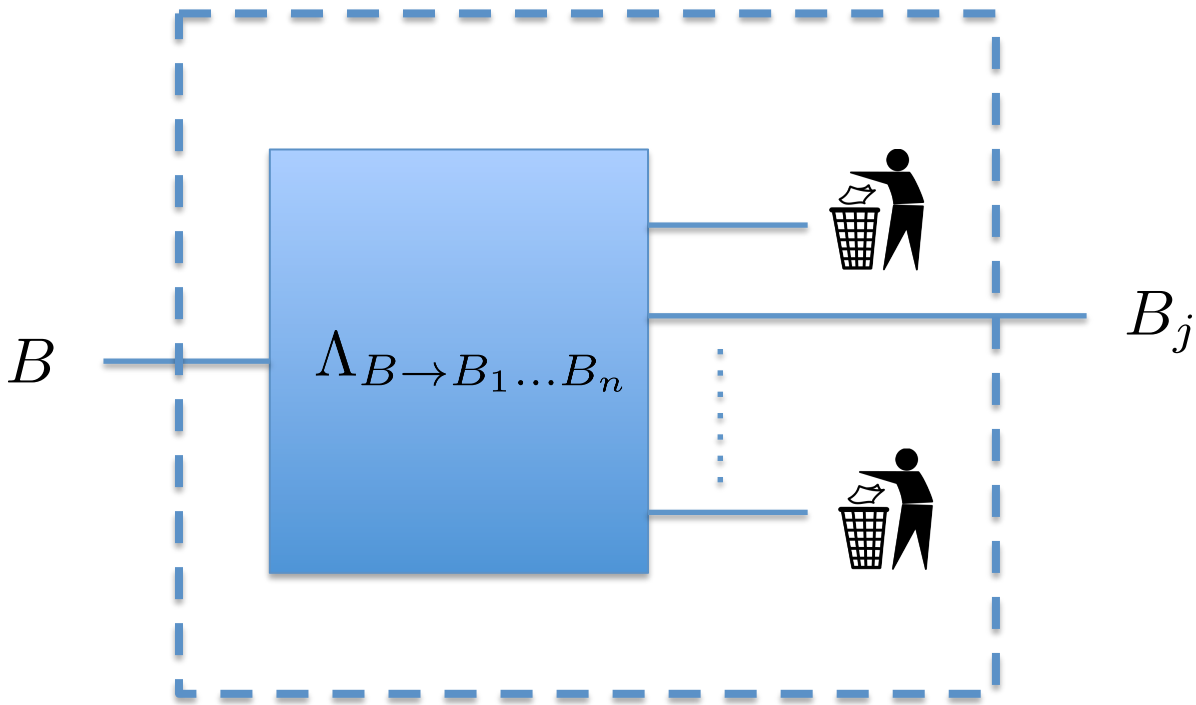


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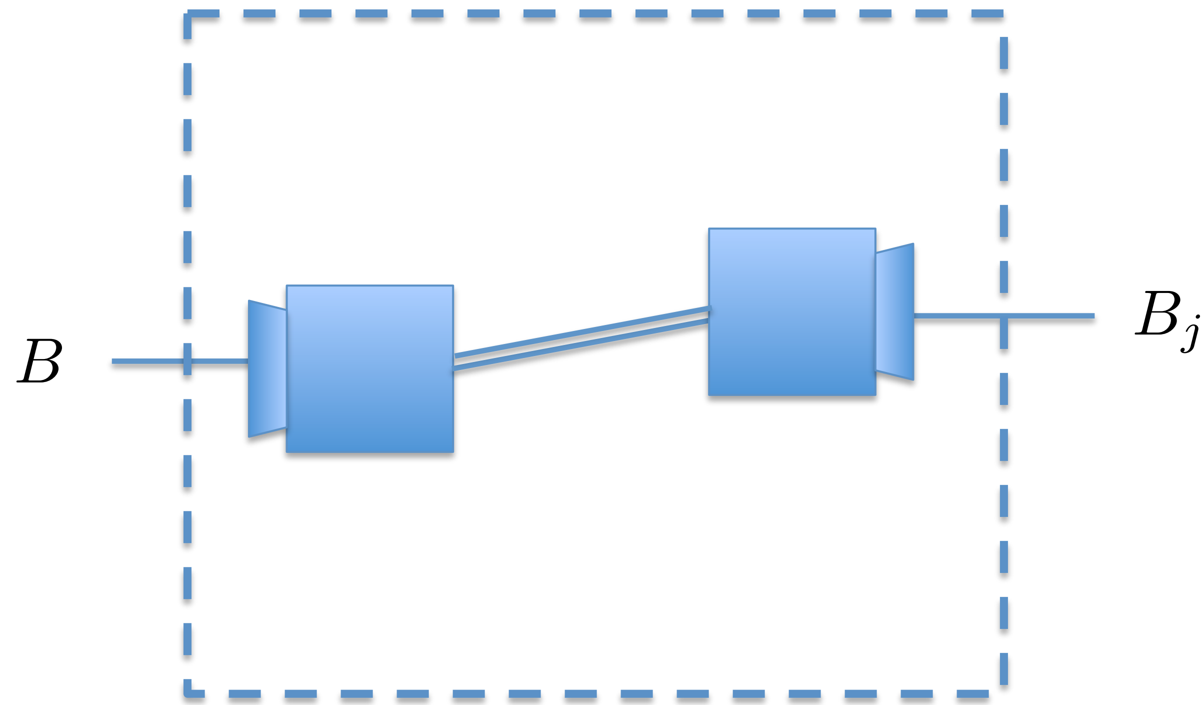



An arbitrarily high fraction of the effective channels are each close to a measure-and-prepare channel, with the measurement stage the same for all of them

[Brandao, P, Horodecki, arXiv:1310.8640]

# Conclusions

- We have introduced a hierarchy of faithful discord-like quantifiers
- The hierarchy deals with the fundamental task of broadcasting / redistribution of correlations

- The hierarchy converges to the surprisal of measurement recoverability, which deals with the fundamental task of classical storing / transmission of quantum information

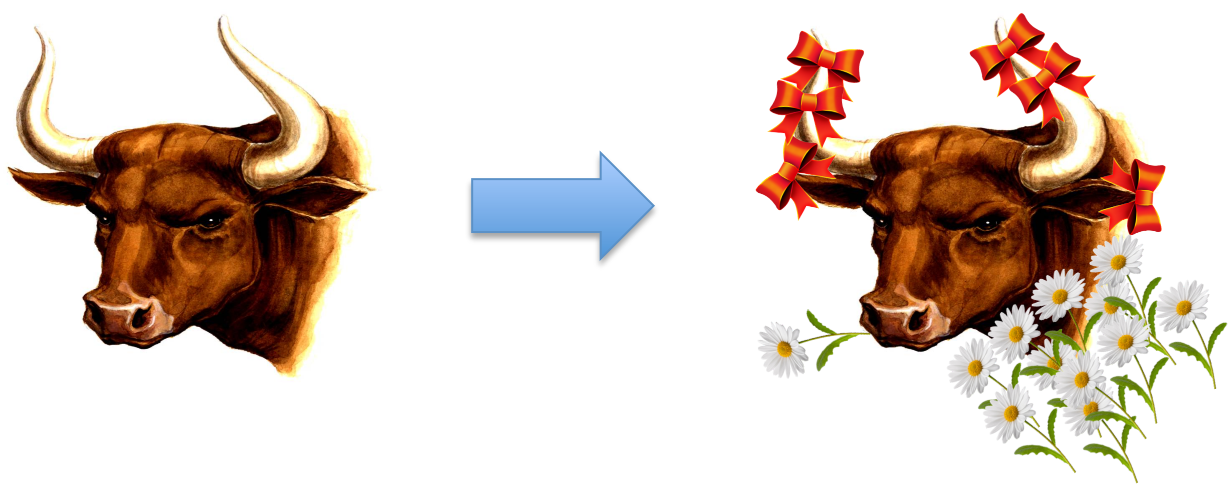
- Each element of the hierarchy corresponds to an SDP

- SDP techniques are ubiquitous in quantum information, in particular to detect entanglement; we extend their use to the study of discord


- Discord itself emerges in considering the average loss of correlations in local broadcasting

# Open questions

- Is a state that is well transmitted through an entanglement-breaking channel close to being quantum-classical?
- What other uses of SDP techniques can be made in the study of discord / quantumness of correlations?
- How can we achieve this?



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# Thanks!

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