Trading resources in quantum Shannon theory

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Based on arXiv:1605.04922, 1206.4886, 1105.0119, 1004.0458, 1001.1732, 0901.3038, 0811.4227 (with Bradler, Guha, Hayden, Hsieh, Qi, Touchette) and Chapter 25 of arXiv:1106.1445

CEQIP 2016, June 17, Valtice, Czech Republic

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- A priori, this question might seem challenging, but there is a surprisingly simple answer for several channels of interest:

  Just combine a single protocol with teleportation, super-dense coding, and entanglement distribution

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- Let $[c \to c]$ denote a noiseless classical bit channel from Alice (sender) to Bob (receiver), which performs the following mapping on a qubit density operator

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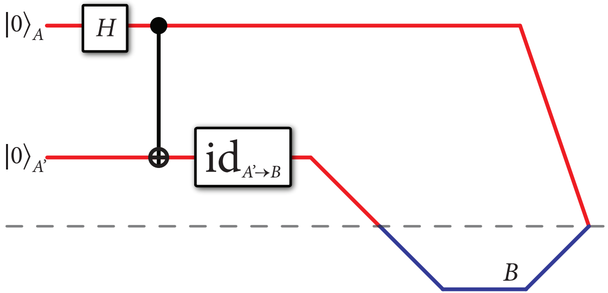
- Entanglement distribution, super-dense coding, and teleportation are non-trivial protocols for combining these resources

Entanglement distribution



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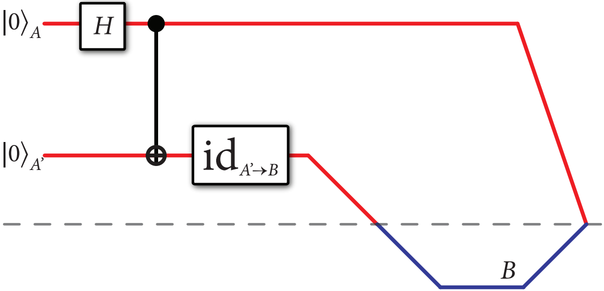

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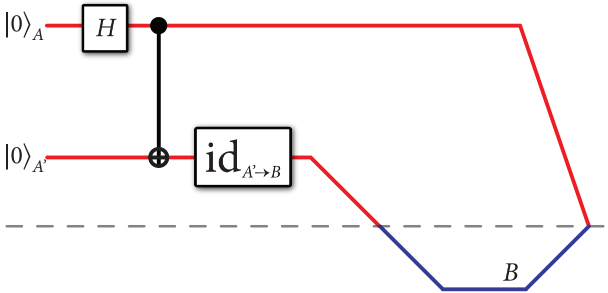

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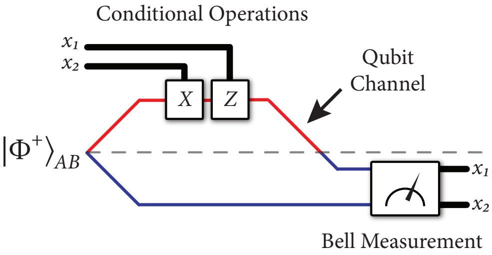

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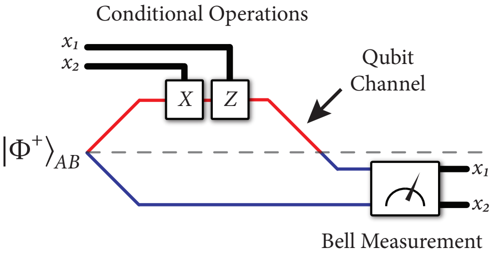

- Alice and Bob share an ebit. Alice would like to transmit two classical bits $x_1 x_2$ to Bob. She performs a Pauli rotation conditioned on $x_1 x_2$ and sends her share of the ebit over a noiseless qubit channel. Bob then performs a Bell measurement to get $x_1 x_2$.

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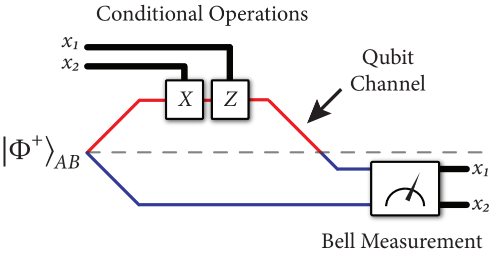

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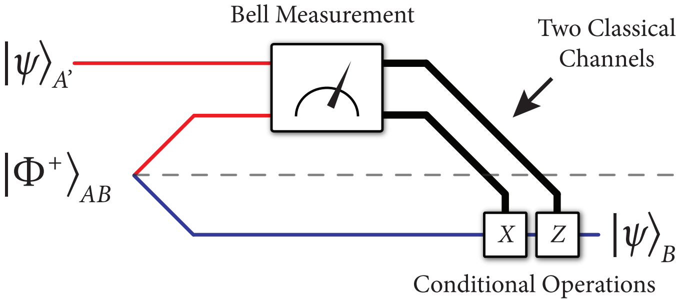
- Resource inequality: $[q \to q] + [qq] \geq 2[c \to c]$

Teleportation [BBC+93]



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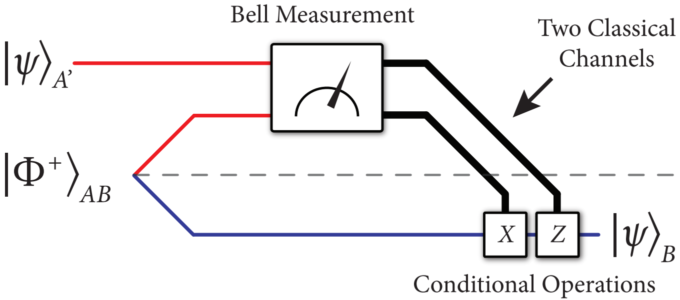

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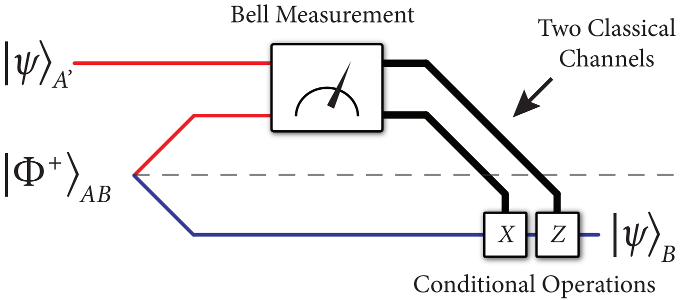

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- All achievable rate triples are then given by

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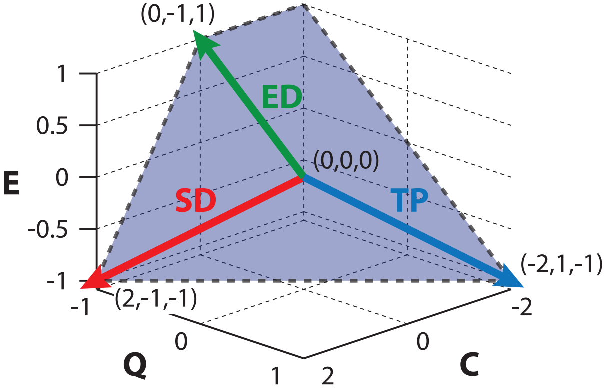
- Writing as a matrix equation, inverting, and applying constraints $\alpha, \beta, \gamma \geq 0$ gives the following achievable rate region:

$$\begin{aligned} C + Q + E &\leq 0, \\ Q + E &\leq 0, \\ C + 2Q &\leq 0. \end{aligned}$$

Unit resource capacity region [HW10]



The unit resource capacity region is $C + Q + E \leq 0$, $Q + E \leq 0$, $C + 2Q \leq 0$ and is provably optimal.

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- That is, what are the rates C_{out}, Q_{out}, E_{out}, C_{in}, Q_{in}, $E_{\text{in}} \geq 0$ achievable in the following resource inequality?

$$\langle \mathcal{N} \rangle + C_{\text{in}}[c \to c] + Q_{\text{in}}[q \to q] + E_{\text{in}}[qq] \\ \geq C_{\text{out}}[c \to c] + Q_{\text{out}}[q \to q] + E_{\text{out}}[qq]$$

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- The union of all achievable rate triples $(C_{\text{out}} - C_{\text{in}}, Q_{\text{out}} - Q_{\text{in}}, E_{\text{out}} - E_{\text{in}})$ is called the quantum dynamic capacity region.

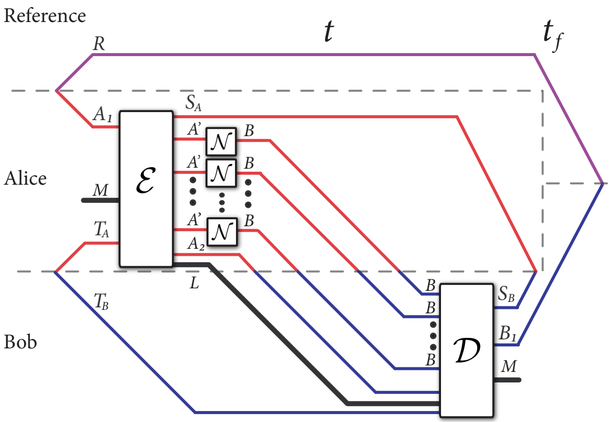

Figure: The most general protocol for generating classical communication, quantum communication, and entanglement with the help of the same respective resources and many uses of a quantum channel.

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- Given a tripartite density operator ρ_{ABC}, the conditional mutual information is defined as

$$I(A;B|C)_\rho = H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho$$

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Define the state-dependent region $\mathcal{C}^{(1)}_{\text{CQE},\sigma}(\mathcal{N})$ as the set of all rates C, Q, and E, such that

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The above entropic quantities are with respect to a classical–quantum state σ_{XAB}, where

$$\sigma_{XAB} \equiv \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}_{A'\to B}(\phi^x_{AA'}),$$

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It is implicit that one should consider states on A'^k instead of A' when taking the regularization.

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where σ_{XABE} is a state of the following form:

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- Combine this with the unit protocols of teleportation, super-dense coding, and entanglement distribution

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- Combining the protocols gives the following set of achievable rates:

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} I(X;B)_\sigma \\ \frac{1}{2} I(A;B|X)_\sigma \\ -\frac{1}{2} I(A;E|X)_\sigma \end{bmatrix},$$

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  where α, β, $\gamma \geq 0$.

- Inverting the matrix equation, applying the constraints α, β, $\gamma \geq 0$, and using entropy identities gives the following region:

$$\begin{aligned} C + 2Q &\leq I(AX;B)_\sigma, \\ Q + E &\leq I(A\rangle BX)_\sigma, \\ C + Q + E &\leq I(X;B)_\sigma + I(A\rangle BX)_\sigma, \end{aligned}$$

  which establishes the achievability part.

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- Erasure channel is defined as follows:

$$\mathcal{N}^{\varepsilon}(\rho) = (1 - \varepsilon)\,\rho + \varepsilon|e\rangle\langle e|,$$

where ρ is a d-dimensional input state, $|e\rangle$ is an erasure flag state orthogonal to all inputs (so that the output space has dimension $d + 1$), and $\varepsilon \in [0, 1]$ is the erasure probability.

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- Let $\mathcal{N}^{\varepsilon}$ be a quantum erasure channel with $\varepsilon \in [0,1/2]$. Then the quantum dynamic capacity region $\mathcal{C}_{\mathrm{CQE}}(\mathcal{N}^{\varepsilon})$ is equal to the union of the following regions, obtained by varying $\lambda \in [0,1]$:

$$C + 2Q \leq (1-\varepsilon)\,(1+\lambda)\log d,$$
$$Q + E \leq (1-2\varepsilon)\,\lambda \log d,$$
$$C + Q + E \leq (1-\varepsilon-\varepsilon\lambda)\log d.$$

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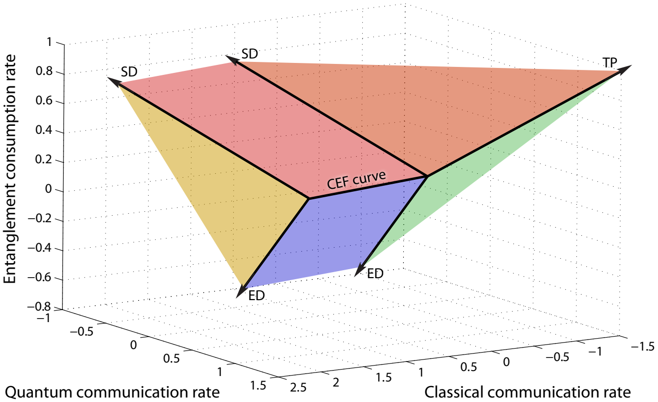

Figure: The quantum dynamic capacity region for the (qubit) quantum erasure channel with $\varepsilon = 1/4$. The plot demonstrates that time-sharing is optimal.

Example: Qubit dephasing channel

The dynamic capacity region $\mathcal{C}_{\text{CQE}}(\overline{\Delta}_p)$ of a dephasing channel with dephasing parameter $p \in [0,1]$ is the set of all C, Q, and E such that

$$C + 2Q \leq 1 + h_2(\nu) - h_2(\gamma(\nu, p)),$$
$$Q + E \leq h_2(\nu) - h_2(\gamma(\nu, p)),$$
$$C + Q + E \leq 1 - h_2(\gamma(\nu, p)),$$

where $\nu \in [0, 1/2]$, h_2 is the binary entropy function, and

$$\gamma(\nu, p) \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 16 \cdot \frac{p}{2}\left(1 - \frac{p}{2}\right)\nu(1-\nu)}.$$

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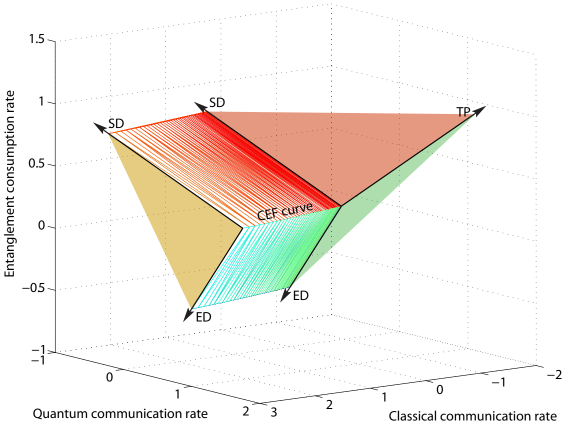

Figure: A plot of the dynamic capacity region for a qubit dephasing channel with dephasing parameter $p = 0.2$. Slight improvement over time-sharing.

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- Pure-loss channel is defined from the following input-output relation:

$$\hat{a} \quad \rightarrow \quad \hat{b} = \sqrt{\eta}\ \hat{a} + \sqrt{1-\eta}\ \hat{e},$$

where $\hat{a}$ is the input annihilation operator for the sender, $\hat{e}$ is the input annihilation operator for the environment, and $\eta \in [0,1]$ is the transmissivity of the channel.

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- Place a photon number constraint on the input mode to the channel, such that the mean number of photons at the input cannot be greater than $N_S \in [0, \infty)$.

Example: Pure-loss bosonic channel [WHG12]

Build trade-off codes from an ensemble of the following form:

$$\left\{ p_{(1-\lambda)N_S}(\alpha), D_{A'}(\alpha)|\psi_{\text{TMS}}(\lambda)\rangle_{AA'} \right\},$$

where $\alpha \in \mathbb{C}$,

$$p_{(1-\lambda)N_S}(\alpha) \equiv \frac{1}{\pi\,(1-\lambda)\,N_S} \exp\left\{ -\left|\alpha\right|^2 / \left[(1-\lambda)\,N_S\right] \right\},$$

$\lambda \in [0,1]$ is a photon-number-sharing parameter, $D_{A'}(\alpha)$ is a "displacement" unitary operator acting on system A', and $|\psi_{\text{TMS}}(\lambda)\rangle_{AA'}$ is a "two-mode squeezed" (TMS) state:

$$|\psi_{\text{TMS}}(\lambda)\rangle_{AA'} \equiv \sum_{n=0}^{\infty} \sqrt{\frac{\left[\lambda N_S\right]^n}{\left[\lambda N_S + 1\right]^{n+1}}}\, |n\rangle_A |n\rangle_{A'},$$

Example: Pure-loss bosonic channel [WHG12]

The quantum dynamic capacity region for a pure-loss bosonic channel with transmissivity $\eta \geq 1/2$ is the union of regions of the form:

$$\begin{aligned} C + 2Q &\leq g(\lambda N_S) + g(\eta N_S) - g((1-\eta)\,\lambda N_S), \\ Q + E &\leq g(\eta\lambda N_S) - g((1-\eta)\,\lambda N_S), \\ C + Q + E &\leq g(\eta N_S) - g((1-\eta)\,\lambda N_S), \end{aligned}$$

where $\lambda \in [0, 1]$ is a photon-number-sharing parameter and $g(N)$ is the entropy of a thermal state with mean photon number N.

The above holds provided that an unsolved multi-mode minimum-output entropy conjecture is true (see next talk of de Palma for solution of the single-mode version of this conjecture). The region is still achievable if $\eta < 1/2$.

Example: Pure-loss bosonic channel [WHG12]

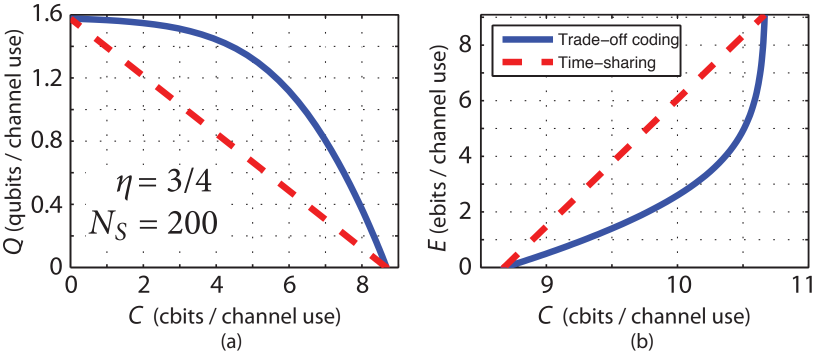

Figure: Suppose channel transmits on average 3/4 of the photons to the receiver, while losing the other 1/4 en route. Take mean photon budget of about 200 photons per channel use at the transmitter. (a) classical–quantum trade-off, (b) classical comm. with rate-limited entanglement consumption. Big gains over time-sharing.

Example: Quantum-limited amplifier channel

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- Amplifier channel is defined from the following input-output relation:

$$\hat{a} \quad \rightarrow \quad \hat{b} = \sqrt{G}\ \hat{a} + \sqrt{G-1}\ \hat{e}^{\dagger},$$

where $\hat{a}$ is the input annihilation operator for the sender, $\hat{e}$ is the input annihilation operator for the environment, and $G \in [1, \infty)$ is the gain of the channel. The channel is *quantum-limited* if the environment is prepared in a vacuum state.

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- Place a photon number constraint on the input mode to the channel, such that the mean number of photons at the input cannot be greater than $N_S \in [0, \infty)$.

Example: Amplifier channel [QW16]

The quantum dynamic capacity region for a quantum-limited amplifier channel with gain $G \geq 1$ is the union of regions of the form:

$$\begin{aligned} C + 2Q &\leq g(\lambda N_S) + g(G N_S + \bar{G}) - g(\bar{G}[\lambda N_S + 1]), \\ Q + E &\leq g(G \lambda N_S + \bar{G}) - g(\bar{G}[\lambda N_S + 1]), \\ C + Q + E &\leq g(G N_S + \bar{G}) - g(\bar{G}[\lambda N_S + 1]), \end{aligned}$$

where $\bar{G} = G - 1$ and $\lambda \in [0, 1]$ is a photon-number-sharing parameter and $g(N)$ is the entropy of a thermal state with mean photon number N. (This holds provided that an unsolved single-mode minimum-output entropy conjecture is true.[1])

[1]arXiv post [QW16] needs an update

Example: Amplifier channel [QW16]

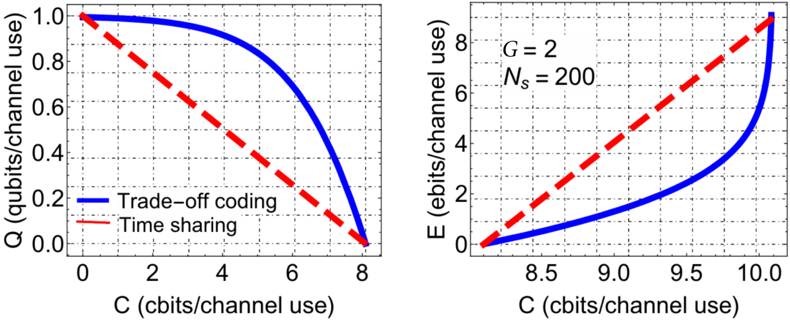

Figure: Suppose channel amplifies with gain $G = 2$ the photons being transmitted to the receiver. Take mean photon budget of about 200 photons per channel use at the transmitter. (a) classical–quantum trade-off, (b) classical comm. with rate-limited entanglement consumption. Big gains over time-sharing.

Conclusion

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- The quantum dynamic capacity theorem characterizes the net rates at which a sender and a receiver can generate classical communication, quantum communication, and entanglement by using a quantum channel many times

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- What if there is feedback from receiver to sender?
- Is there a simple characterization for distillation tasks? For progress, see [HW10]

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