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Bell inequalities for maximally entangled states

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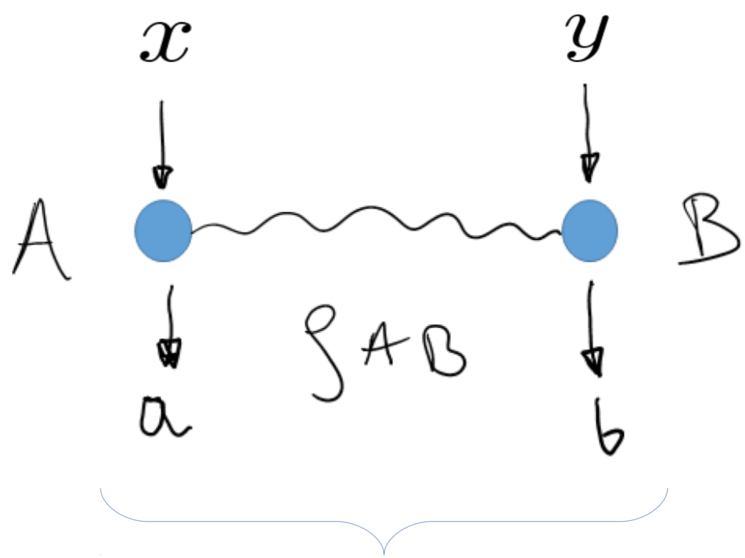
Preliminaries

- **Bell scenario:** consider two parties performing measurements on their local systems



$$p(a,b|x,y)$$

measurement choices

$$x, y = 1, \ldots, m$$

(2,m,d) scenario

outcomes

$$a, b = 0, \ldots, d-1$$

measurement $M = \{M_a\}_a$

$$M^a \geq 0, \quad \sum_a M^a = \mathbb{1} \quad \text{(POVM)}$$

$$M^a M^{a'} = \delta_{a,a'} M^a \quad \text{(PM)}$$

- Correlations are described by a collection of prob. distributions

$$\{p(a,b|x,y)\}_{a,b;x,y}$$

$$p(a,b|x,y) = \mathrm{Tr}[\rho_{AB}(M_x^a \otimes N_y^b)]$$

$$\{p(a,b|x,y)\}$$

correlations/behavior

Preliminaries

Nonlocality and Bell inequalities

- Local/classical correlations

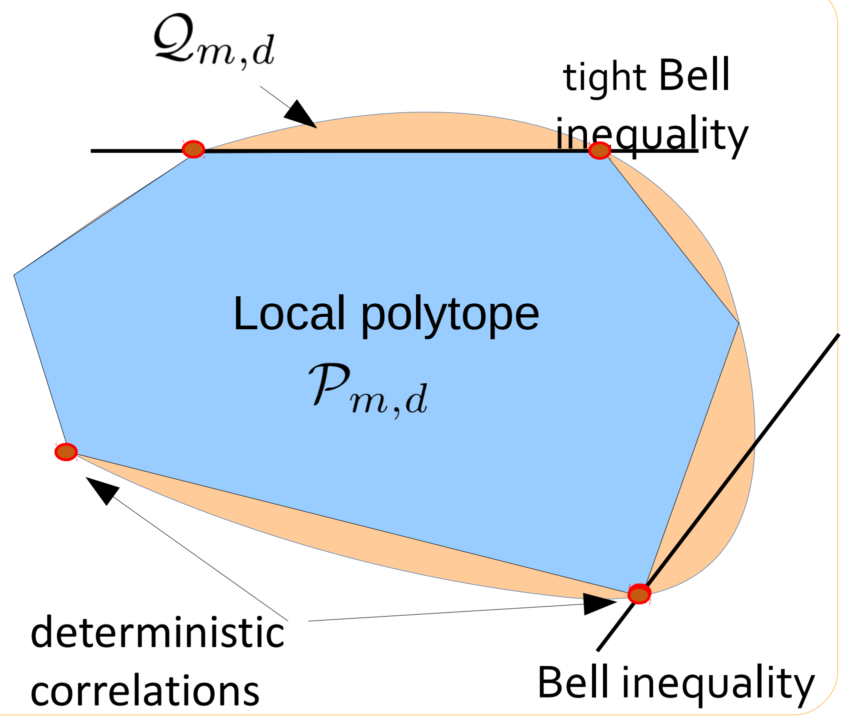
$$p(a,b|x,y)=\sum_{\lambda}p(\lambda)p_{\mathrm{det}}(a|x,\lambda)p_{\mathrm{det}}(b|y,\lambda)$$

$$\forall_{a,b,x,y,\lambda}\qquad p_{\mathrm{det}}(a|x,\lambda),p_{\mathrm{det}}(b|y,\lambda)\in\{0,1\}$$

- Otherwise they are called **nonlocal (nonlocality)**

$$\mathcal{P}_{m,d}\subsetneq\mathcal{Q}_{m,d}$$

[J. S. Bell, Physics **1**, 195 (1964)]



- **Bell inequalities**: Hyperplanes constraining the local set

$$I:=\sum_{a,b,x,y}T^{a,b}_{x,y}\,p(a,b|x,y)\le\beta_C$$

$$(\beta_C=\max_{\mathcal{P}_{m,d}}I)$$

Examples

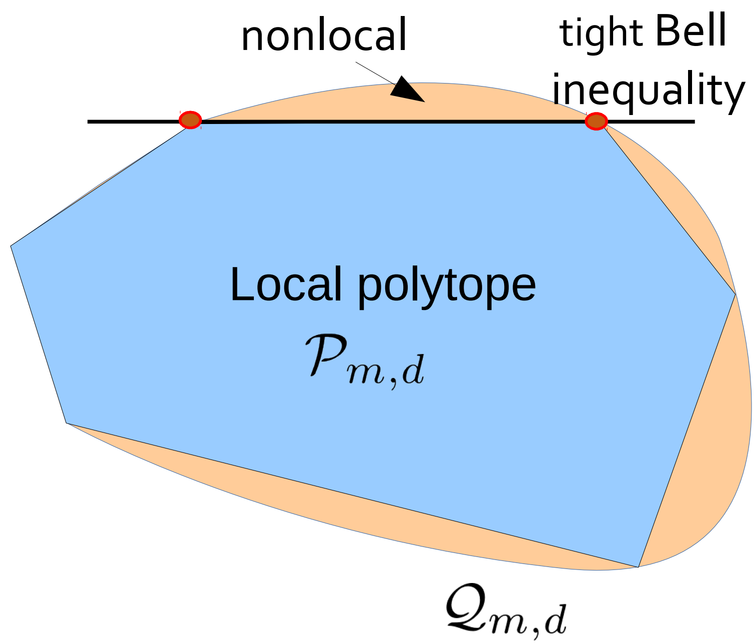
Clauser, Horne, Shimony, Holt (1969);
Collins *et al.* (CGLMP) (2002);
Barrett, Kent, Pironio (BKP) (2006);

Finite number of BI's (facets) enough to fully characterize the local polytope → tight Bell inequalities (convex hull problem)

The problem

- Qualitative and quantitative statements about relevant quantum properties
  - randomness certification, amplification, expansion
    [Pironio *et al.*, Nature (2010); Renner, Colbeck, Nat. Phys. (2012)]
  - bounds on key rates in QKD [Pironio *et al.*, PRX (2013)]
  - dimension witnesses [Brunner *et al.*, PRL (2008)]

$$\beta_Q = \max_{\mathcal{Q}_{m,d}} I$$



- **The problem:** no general class of Bell inequalities for maxent quantum states

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B \in \mathbb{C}^d \otimes \mathbb{C}^d$$

- CHSH (1971) – (2,2,2) scenario
- Son *et al.* (2006) – (2,2,d) scenario
- Ji *et al.* (2008) , Liang *et al.* (2009) – (2,2,3) scenario
  MUB observables

- perfect correlations in any local basis (QKD)
- $\rho_A = \rho_B = \mathbb{1}/d$
  (randomness certification)
- maximizer of entanglement measures

Constructing Bell inequalities

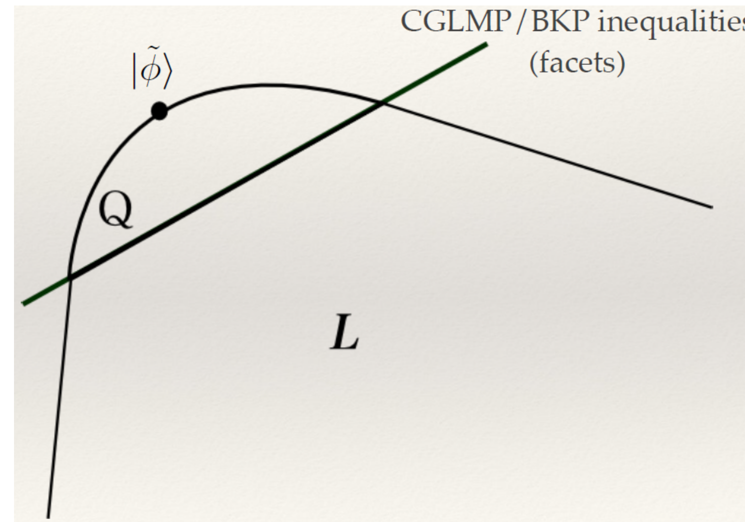
Phys. Rev. Lett. 119, 040402 (2017)

- Consider the Barrett-Kent-Pironio (BKP) Bell expression

[Collins *et al.*, PRL (2002); Barrett *et al.*, PRL (2006)]

$$I_{d,m} := \sum_{k=0}^{\lfloor d/2 \rfloor - 1} (\mathbb{P}_k - \mathbb{Q}_k)$$

$p(a,b|x,y)$



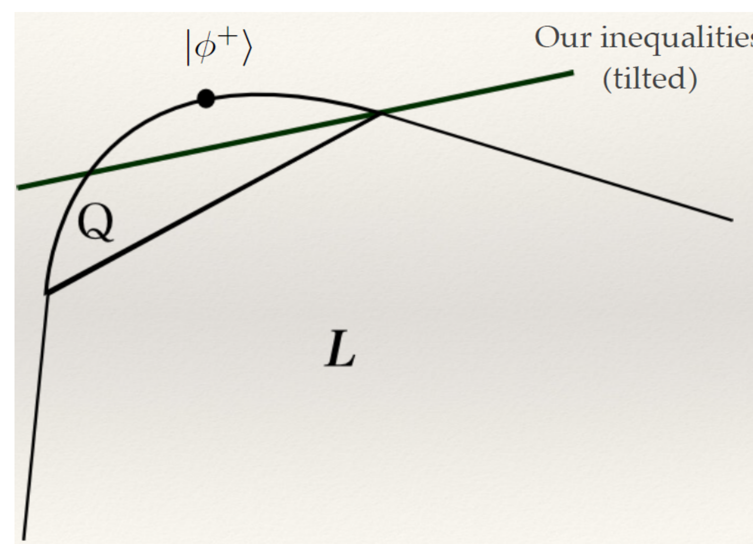
  - facet Bell inequalities in (2,2,d) scenario

[Masanes, QIC (2002)]

  - not maximally violated by $|\psi_d^+\rangle$
  - e.g., for *d=3*

$$|\tilde{\phi}\rangle \sim |00\rangle + \gamma|11\rangle + |22\rangle \qquad \gamma = \tfrac{1}{2}(\sqrt{11} - \sqrt{3})$$

[Acin, Durt, Gisin, QIC (2002); Yang *et al.* (2014)]



- Modify by adding parameters (tilting the inequality)

$$I_{d,m} := \sum_{k=0}^{\lfloor d/2 \rfloor - 1} (\alpha_k \mathbb{P}_k - \beta_k \mathbb{Q}_k)$$

$$\alpha_k, \beta_k \in \mathbb{R}$$

Constructing Bell inequalities

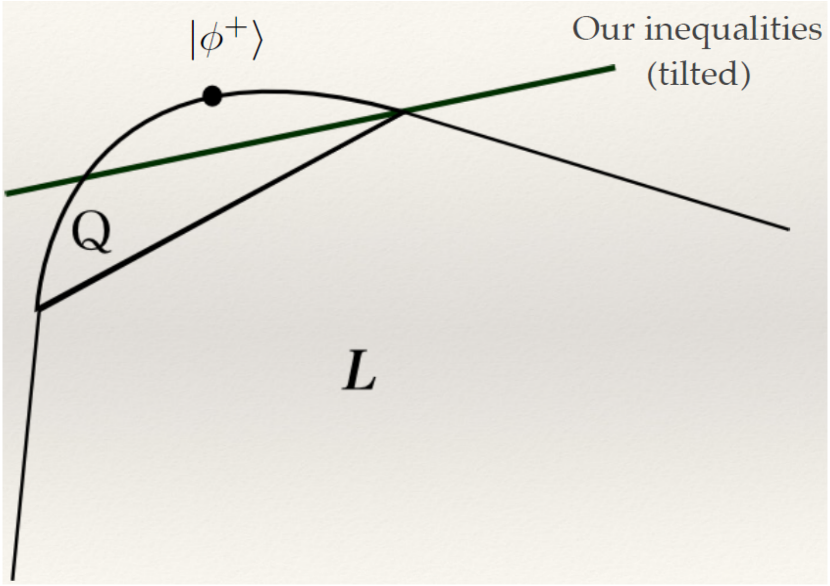
Phys. Rev. Lett. 119, 040402 (2017)

- How to find the coefficients?
  - Fourier transform → correlation form

$$\widetilde{I}_{d,m} := \sum_{i=1}^{m}\sum_{l=0}^{d-1}\left(a_l\langle A_i^l B_i^{d-l}\rangle + a_l^*\langle A_i^l B_{i-1}^{d-l}\rangle\right)$$

$$a_l \equiv a_l(\alpha_k, \beta_k)$$

$$\langle A_x^k B_y^l\rangle = \sum_{a,b=0}^{d-1}\omega^{ak+bl}p(a,b|x,y)$$



A_i, B_i – unitary with eigenvalues $1, \omega, \ldots, \omega^{d-1}$ $\qquad \omega = \exp(2\pi\mathbb{i}/d)$

- "Quantum approach" (CHSH example)

$$I_{\mathrm{CHSH}} = \langle A_0 \otimes (B_0 + B_1)\rangle + \langle A_1 \otimes (B_0 - B_1)\rangle \leq 2$$

$$A_0 \otimes \frac{B_0+B_1}{\sqrt{2}}|\psi_2^+\rangle = \sigma_x \otimes \sigma_x|\psi_2^+\rangle = |\psi_2^+\rangle$$

$$A_1 \otimes \frac{B_0-B_1}{\sqrt{2}}|\psi_2^+\rangle = \sigma_z \otimes \sigma_z|\psi_2^+\rangle = |\psi_2^+\rangle$$

optimal CHSH measurements

$$A_{0/1} = \sigma_{x/z}, \qquad B_{0/1} = \frac{\sigma_x \pm \sigma_z}{\sqrt{2}}$$

$$|\psi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Constructing Bell inequalities

- "Quantum approach"

$$\widetilde{I}_{d,m} := \sum_{i=1}^{m}\sum_{l=1}^{d-1}\left(a_l\langle A_i^l B_i^{d-l}\rangle + a_l^*\langle A_i^l B_{i-1}^{d-l}\rangle\right) = \sum_{i=1}^{m}\sum_{l=1}^{d-1}\langle A_i^l \bar{B}_i^{(l)}\rangle$$

$$a_l \equiv a_l(\alpha_k, \beta_k) \qquad \bar{B}_i^{(l)} = a_l B_i^{d-l} + a_l^* B_{i-1}^{d-l}$$

- we want α_k, β_k so that

$$A_i^l \otimes \bar{B}_i^{(l)}|\psi_+^d\rangle = |\psi_+^d\rangle$$

for all $i = 1, \ldots, m \quad l = 1, \ldots, d-1$

A_i, B_i

CGLMP/BKP

measurements

[Collins *et al.* (CGLMP) (2002);
Barrett, Kent, Pironio (BKP) (2006)]

$$\bar{B}_i^{(l)} = (A_i^l)^*$$

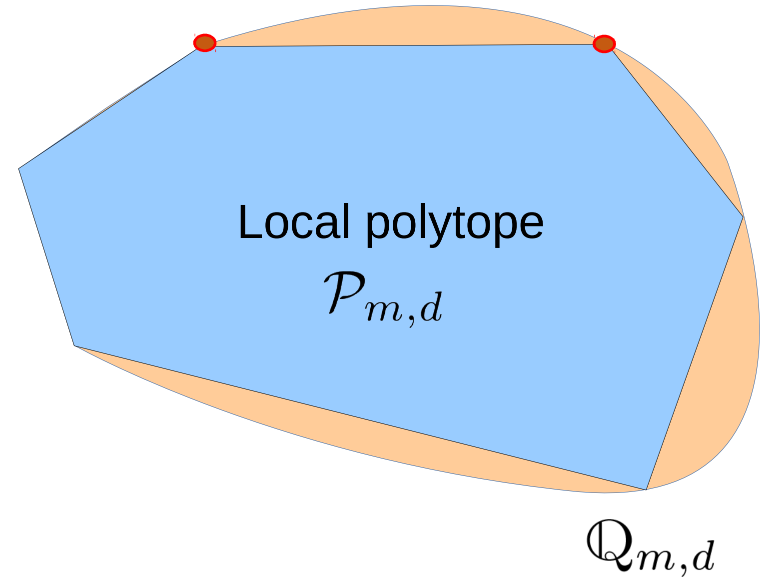
→ α_k, β_k almost uniquely

Full characterization of our Bell inequalities

Phys. Rev. Lett. **119**, 040402 (2017)

- Analytical proof of the maximal quantum value

$$\beta_Q = \max_{\mathcal{Q}_{m,d}} \tilde{I}_{m,d} = m(d-1)$$



- sum of squares decomposition

$$\beta_Q \mathbb{1} - \mathcal{B} = \frac{1}{2}\sum_{i=1}^{m}\sum_{k=1}^{d-1} P_{ik}^{\dagger}P_{ik} + \frac{1}{2}\sum_{i=1}^{m-2}\sum_{k=1}^{d-1} T_{ik}^{\dagger}T_{ik}$$

$$P_{ik} = \mathbb{1} - A_i^k \otimes \bar{B}_i^k \qquad T_{ik} = \mu_{i,k} B_2^{d-k} + \nu_{i,k} B_{i+2}^{d-k} + \tau_{ik} B_{i+3}^{d-k}$$

some coefficients

- quantum realization $|\psi_d^+\rangle$ and optimal CGLMP/BKP measurements

- **Example**: the CHSH inequality

$$2\sqrt{2}\,\mathbb{1} - \mathcal{B}_{\text{CHSH}} = \frac{1}{\sqrt{2}}\left[\left(\mathbb{1} - A_0 \otimes \frac{B_0+B_1}{\sqrt{2}}\right)^2 + \left(\mathbb{1} - A_1 \otimes \frac{B_0-B_1}{\sqrt{2}}\right)^2\right]$$

for all dichotomic A_i, B_i

Full characterization of our Bell inequalities

- Analytical computation of the classical bound

[Barrett *et al.*, PRL (2006)]

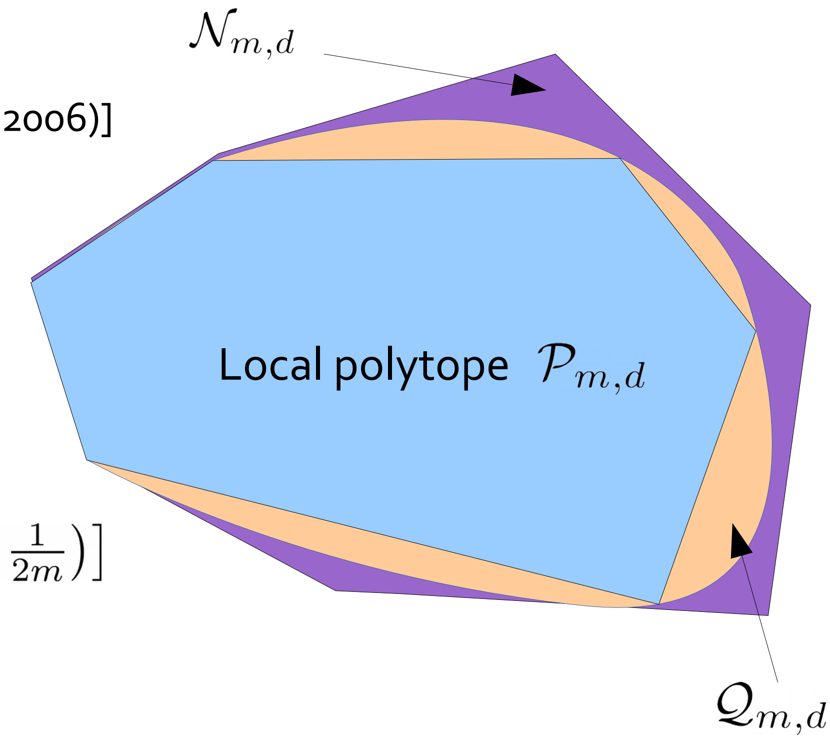
$$\beta_C = \max_{\mathcal{P}_{m,d}} \tilde{I}_{m,d}$$

- The maximal nonsignaling value

$$\beta_{NS} = \max_{\mathcal{N}_{m,d}} \tilde{I}_{m,d} = m \tan\left(\frac{\pi}{2m}\right) g(0) - m$$

$$g(x) = \cot\left[\frac{\pi}{d}\left(x + \frac{1}{2m}\right)\right]$$

- Asymptotic properties of β_Q/β_C and β_{NS}/β_Q



- Special cases

m=2 – Bell inequalities considered by Son, Lee and Kim

[Phys. Rev. Lett. (2006); J. de Vicente, Phys. Rev. A (2015)]

d=2 – the chained Bell inequalities

[Pearle, 1970; Braunstein, C. Caves, 1990; Wehner (2006)]

$$\langle A_0 B_0 \rangle + \langle A_0 B_m \rangle + \sum_{i=1}^{m} \left(\langle A_i B_i \rangle + \langle A_i B_{i-1} \rangle \right) \leq 2(m-1)$$

Applications

[Mayers, Yao, 2004]

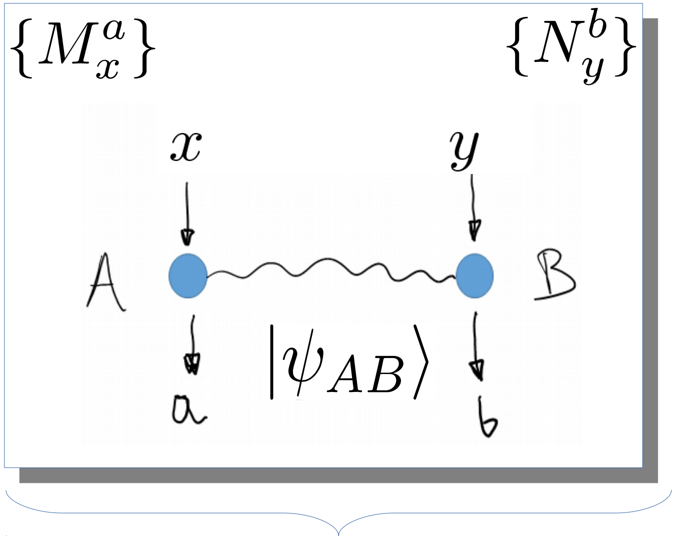
- Device-independent applications
  - Quantum key distribution – better lower bounds on key rates
  - Self-testing of maximally entangled states for d>2

- The idea of self-testing (DI certification)
  - $\{p(a,b|x,y)\}_{a,b;x,y}$
  - or violation of some Bell inequality

$$\sum_{a,b,x,y} T_{x,y}^{a,b}\, p(a,b|x,y) = \beta$$

  - **deduce properties of** $|\psi_{AB}\rangle$ **and** $\{M_x^a\},\ \{N_y^b\}$

Quantum device



$\{p(a,b|x,y)\}_{a,b;x,y}$

Self-testing maxent state

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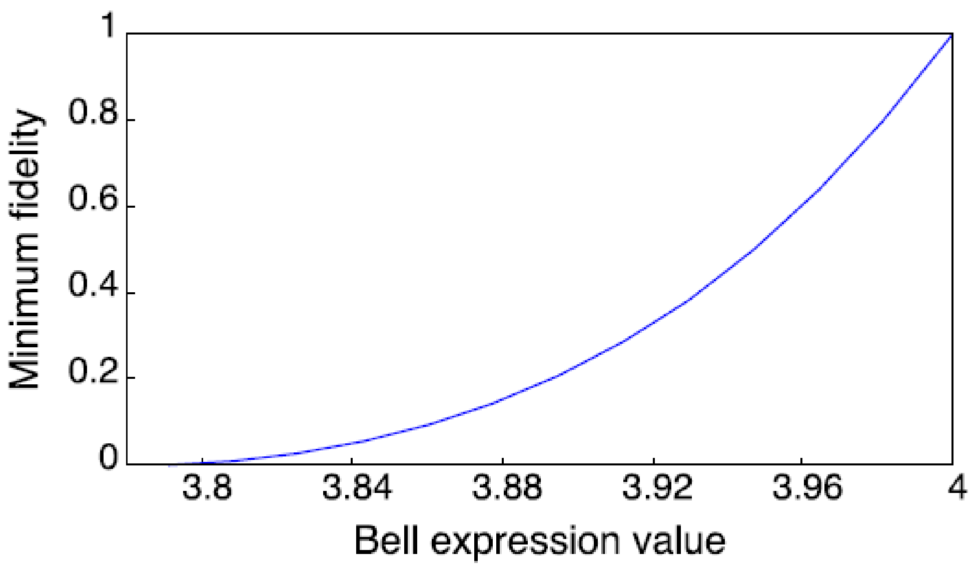
- Numerical for *d=3 (m=2)* [Yang *et al.*, 2014; Wu *et al.*, 2014]



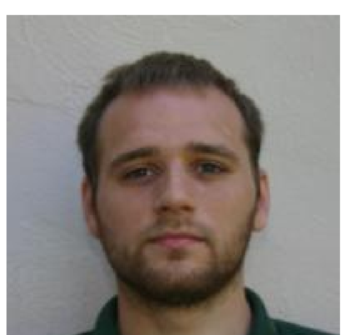
$$\min_{\rho_{AB}} \langle \psi_3^+ | \rho_{AB} | \psi_3^+ \rangle \geq f(\beta)$$

$$\beta_C \simeq 3$$

maximal quantum value

- Rigorous analytical proof for *d=3,4,5*

see Jędrek's talk!



An approach based on the CHSH Bell inequality [Yang, Navascués, 2012]

Generalization to *N* parties

- Bell inequalities maximally violated by *N* qu*d*it GHZ states

$$|\mathrm{GHZ}_{N,d}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle^{\otimes N}$$

- Consider the Bell expression (*N*=3) [Aolita *et al.*, PRL (2012); Bancal *et al. JPA* (2012)]

$$I_{N,d,m} := \sum_{k=0}^{\lfloor d/2 \rfloor -1} (\alpha_k \mathbb{P}_k - \beta_k \mathbb{Q}_k)$$

$$p(a,b,c|x,y,z)$$

$\alpha_k, \beta_k \in \mathbb{R}$ free parameters

$$\alpha_k = \beta_k = 1 - \frac{2k}{d-1}$$

- Go to the correlation picture (the Fourier transform)

$$\tilde{I}_{3,M,d} = \frac{1}{d} \sum_{x,y=1}^{M} \sum_{k=0}^{d-1} \langle \bar{A}_x^k B_{x+y-1}^{-k} C_y^k \rangle,$$

A_x, B_y, C_z – unitary observables with eigenvalues $1, \omega, \ldots, \omega^{d-1}$

$$\bar{A}_x^k = a_k(\vec{\alpha}, \vec{\beta}) A_x^k + a_k^*(\vec{\alpha}, \vec{\beta}) A_{x+1}^k$$

Generalization to N parties

- Solve a system of equations

$$\bar{A}_x^k \otimes B_{x+y-1}^{-k} \otimes C_y^k |\mathrm{GHZ}_{3,d}\rangle = |\mathrm{GHZ}_{3,d}\rangle \qquad \text{for all } x, y, k$$

for fixed A_x, B_y, C_z → α_k, β_k almost uniquely

- Characterization
  - Maximal quantum violation (SOS similar to the bipartite one)

$$\beta_{N,m,d}^{Q} = m^{N-1}(d-1) = m^{N-2}\beta_{2,m,d}^{Q}$$

  - The maximal nonsignaling value

$$\beta_{N,m,d}^{\mathrm{NS}} = m^{N-2}\beta_{2,m,d}^{\mathrm{NS}}$$

  - Maximal classical value

$$\beta_{N,m,d}^{\mathrm{C}} = ? \qquad \beta_{N,m,d}^{\mathrm{C}} \neq m^{N-2}\beta_{2,m,d}^{\mathrm{C}}$$

Conclusion/Outlook

- A class of Bell inequalities for maxent states
- Analytical computation of all relevant quantities $\beta_C,\ \beta_Q,\ \beta_{NS}$
- Numerical self-testing for *d=3* → for the rigorous analytical proof see Jed's talk
- Generalization to the GHZ states

- Self-testing of maxent states for any *d*?
- Other Bell inequalities max. violated by the max. entangled states?

  modification of the CHSH-*d* Bell inequality [Buhrman, Massar (2005); Ji et al. (2008); Bavarian, Shor (2013)] → see Jed's talk

- Bell inequalities for other entangled quantum states of local dimension >2

  partially entangled bipartite states

  multipartite states such as AME states or graph states