Fundamental limitations on the capacities of bipartite quantum interactions

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Unitary Bipartite Quantum Interactions

- Unitary bipartite interactions fundamental features in many areas of quantum physics.
- Interaction, described by Hamiltonian, between quantum system and heat bath.
- Measurement: Interaction of a quantum system with a measurement device.
- Noisy evolution: Interaction of a quantum system with the environment.
- Bipartite quantum gates in quantum computation and other quantum protocols.
- Corresponding unitary CPTP map

$$\mathcal{U} : A'B' \to AB.$$

- Can increase or decrease entanglement between Alice and Bob.

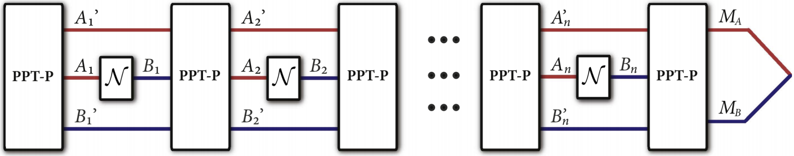
Noisy Bipartite Quantum Interactions

- Tripartite unitary between two systems and environment.
- Noisy quantum gates.
- Interaction of two systems with a heat bath.
- Joint measurement of two quantum systems, e.g. for teleportation.
- Corresponding CPTP map: Bidirectional channel

$$\mathcal{N} : A'B' \to AB.$$

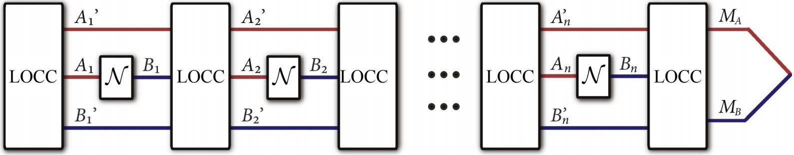
- Can increase or decrease entanglement between Alice and Bob.

Entanglement Generation



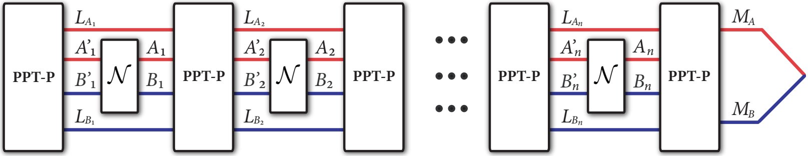
- LOCC-assisted quantum capacity: $\mathcal{Q}^{\leftrightarrow}(\mathcal{N}_{A\to B})$.
- LOCC $\subset$ PPT: $\mathcal{Q}^{\leftrightarrow}(\mathcal{N}_{A\to B}) \leq \mathcal{Q}^{\leftrightarrow}_{PPT}(\mathcal{N}_{A\to B})$.
- [Berta, Wilde. NJP 20.5 (2018): 053044]: PPT-assisted quantum capacity bounded from above by max-Rains information: $\mathcal{Q}^{\leftrightarrow}_{PPT}(\mathcal{N}_{A\to B}) \leq R_{\max}(\mathcal{N}_{A\to B})$.
- $R_{\max}(\rho_{AB}) = \min_{\sigma_{AB}\in \mathrm{PPT}'} D_{\max}(\rho||\sigma)$, where
  $\mathrm{PPT}' = \{\sigma_{AB} \geq 0 : \|\sigma^{\Gamma}\|_1 \leq 1\}$,
  $D_{\max}(\rho||\sigma) = \inf\left\{\lambda \in \mathbb{R}^+ : \rho \leq 2^{\lambda}\sigma\right\}$.
- [Wang, Fang, Duan. arXiv:1709.00200]: Max-Rains information computable by SDP.

Secret Key Generation



- [Horodecki$^{\otimes 3}$, Oppenheim IEEE, 55(4), 1898]: Private states $\gamma^d = 1/d \sum_{ij} |ii\rangle\langle jj| \otimes U^{(i)} \sigma U^{(j)\dagger}$. Can be arbitrarily close to PPT states.
- Private capacity $\mathcal{P}^{\leftrightarrow}(\mathcal{N}_{A\to B}) \geq \mathcal{Q}^{\leftrightarrow}(\mathcal{N}_{A\to B})$.
- $\mathcal{Q}^{\leftrightarrow}(\mathcal{N}_{A\to B}) = 0$, $\mathcal{P}^{\leftrightarrow}(\mathcal{N}_{A\to B}) > 0$ possible.
- Upper bounds: $E_{sq}(\mathcal{N}_{A\to B}), E_{\max}(\mathcal{N}_{A\to B}), E_R(\mathcal{N}_{A\to B})$.
- $E_{\max}(\rho_{AB}) = \min_{\sigma_{AB}\in \text{SEP}} D_{\max}(\rho||\sigma)$.

Entanglement Generation via Bipartite Interactions



- PPT-assisted bidirectional quantum capacity

$$\mathcal{Q}_{PPT}^{2\to 2}(\mathcal{N}_{A'B'\to AB}).$$

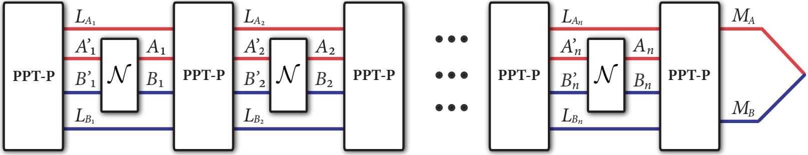
- Bidirectional max-Rains Information

$$R_{\max}^{2\to 2}(\mathcal{N}_{A'B'\to AB}) = \log \Gamma^{2\to 2}(\mathcal{N}_{A'B'\to AB}),$$

where

$$\begin{aligned}\Gamma^{2\to 2}(\mathcal{N}_{A'B'\to AB}) = \text{minimize} \quad & \|\mathrm{Tr}_{AB}\{V_{S_AABS_B} + Y_{S_AABS_B}\}\|_\infty \\ \text{subject to} \quad & V_{S_AABS_B}, Y_{S_AABS_B} \geq 0, \\ & (V_{S_AABS_B} - Y_{S_AABS_B})^{\Gamma_{BS_B}} \geq J^{\mathcal{N}}_{S_AABS_B}.\end{aligned}$$

Entanglement Generation via Bipartite Interactions



Theorem

The PPT-assisted bidirectional quantum capacity of a bidirectional channel $\mathcal{N}_{A'B'\to AB}$ is bounded from above by its bidirectional max-Rains information:

$$Q^{2\to 2}_{PPT}(\mathcal{N}_{A'B'\to AB}) \leq R^{2\to 2}_{\max}(\mathcal{N}_{A'B'\to AB}),$$

and this upper bound is in fact a strong converse bound.

Sketch of proof

- Main observation: $R_{\max}^{2\to 2}$ cannot be enhanced by amortization:

$$R_{\max}(S_A A; BS_B)_\sigma \leq R_{\max}(S_A A'; B' S_B)_\rho + R_{\max}^{2\to 2}(\mathcal{N}_{A'B'\to AB}),$$

  where $\sigma_{S_A ABS_B} = \mathcal{N}_{A'B'\to AB}(\rho_{S_A A' B' S_B})$.
- Successive application to every channel:

$$R_{\max}(M_A; M_B)_\omega \leq n R_{\max}^{2\to 2}(\mathcal{N}),$$

  where $\omega_{M_A M_B}$ is the final state.
- Look at success probabilities in entanglement test: $\operatorname{Tr}\{\Phi\omega\} \geq 1-\varepsilon$. By Rains, $\operatorname{Tr}\{\Phi\sigma\} \leq \frac{1}{M}$ for $\sigma \in \mathrm{PPT}$. Hence $R_{\max}(M_A; M_B)_\omega \geq \log[(1-\varepsilon)M]$.
- Hence,

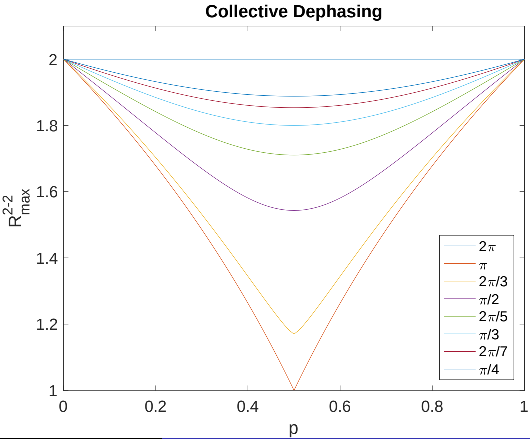
$$\frac{1}{n}\log M \leq R_{\max}^{2\to 2}(\mathcal{N}) + \frac{1}{n}\log\left(\frac{1}{1-\varepsilon}\right).$$

- Strong converse: Solve for ε.

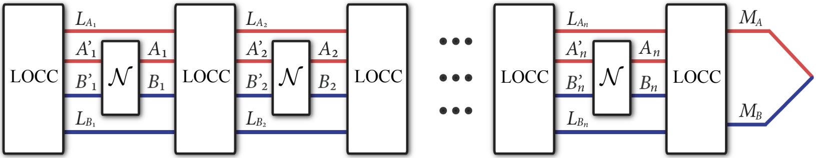
Example: Collective Dephasing

- Collective dephasing: $|00\rangle \to |00\rangle$, $|01\rangle \to e^{i\phi}|01\rangle$, $|10\rangle \to e^{i\phi}|10\rangle$, $|11\rangle \to e^{2i\phi}|11\rangle$.
- Swap operator $S = \sum_{ij} |ij\rangle\langle ji|$ and collective dephasing:

$$\mathcal{N}_{A'B'\to AB}(\rho) = pS\rho S^\dagger + (1-p)U^\phi S\rho S^\dagger U^{\phi\dagger}$$



Secret Key Generation via Bipartite Interactions



- LOCC-assisted bidirectional private capacity

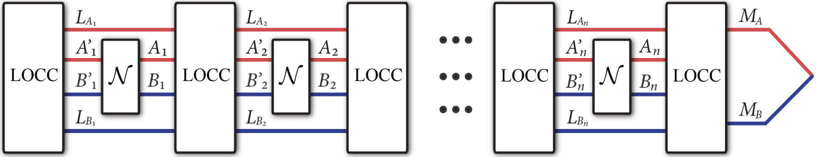
$$\mathcal{P}^{2\to 2}_{LOCC}(\mathcal{N}_{A'B'\to AB}).$$

- Bidirectional max-relative entropy of entanglement:

$$E^{2\to 2}_{\max}(\mathcal{N}_{A'B'\to AB}) = \sup_{\psi_{S_A A'}\otimes\varphi_{B'S_B}} E_{\max}(S_A A; BS_B)_\omega,$$

where $\omega_{S_A ABS_B} = \mathcal{N}_{A'B'\to AB}(\psi_{S_A A'} \otimes \varphi_{B'S_B})$ and $\psi_{S_A A'}$ and $\varphi_{B'S_B}$ are pure bipartite states such that $S_A \simeq A', S_B \simeq B'$.

Secret Key Generation via Bipartite Interactions



Theorem

The LOCC-assisted bidirectional quantum capacity of a bidirectional channel $\mathcal{N}_{A'B'\to AB}$ is bounded from above by its bidirectional max-relative entropy of entanglement:

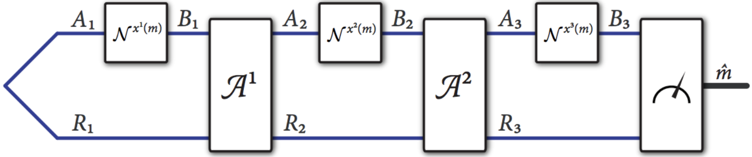
$$P^{2\to2}_{LOCC}(\mathcal{N}_{A'B'\to AB}) \leq E^{2\to2}_{\max}(\mathcal{N}_{A'B'\to AB}),$$

and this upper bound is in fact a strong converse bound.

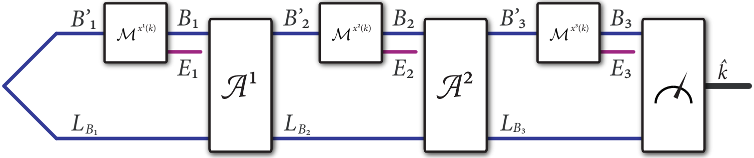
- Proof analogously to quantum capacity, using that $E^{2\to2}_{\max}$

Application: Private Reading

- Excursion: Quantum Reading [Pirandola PRL 106.9 (2011): 090504, Das, Wilde arXiv:1703.03706].
- Given classical storage device, where information can be encoded into a sequence of channels taken from a memory cell $S_X = \{\mathcal{N}^x_{B' \to B}\}_{x \in X}$.
- Example: CD with bit stored into two reflectivities r_0, r_1: Memory cell consisting of two attenuator channels $\{\mathcal{E}^{r_0}, \mathcal{E}^{r_1}\}$.
- Alice encodes $m \in \mathcal{M}$ into codewords $(x_1(m), ..., x_n(m))$, and sets the device to $\left(\mathcal{N}^{x_1(m)}_{B' \to B}, ..., \mathcal{N}^{x_n(m)}_{B' \to B}\right)$.
- Bob can enter quantum states and do channel discrimination to learn m. Generally adaptive strategy.



Application: Private Reading



- Private version: Eve present when Bob performs the readout: Wiretap memory cell $\bar{S}_X = \{\mathcal{N}^x_{B'\to BE}\}_{x\in X}$.
- Special case: Isometric wiretap memory cell $\bar{S}^{\text{iso}}_X$.
- Strong converse upper bound on private capacity by considering quantum key generation by *coherent* version of writing and reading protocol.

$$\mathcal{P}\left(\bar{S}^{\text{iso}}_X\right) \leq \mathcal{P}^{2\to 2}_{LOCC}\left(\sum_{x\in X} |x\rangle\langle x|_{A'\to A} \otimes \mathcal{N}^x_{B'\to B}\right),$$

where $\mathcal{N}^x(\cdot) = \text{Tr}_E(U^x_{B'\to BE}(\cdot)U^{x\dagger}_{B'\to BE})$.

- $\mathcal{N}_{A'B'\to AB}$ PPT-simulable/ teleportation-simulable if there exists a PPT-preserving (or LOCC) channel $\mathcal{T}_{S_A A' B' S_B \to AB}$, such that

$$\mathcal{N}_{A'B'\to AB}\left(\rho_{A'B'}\right) = \mathcal{T}_{S_A A' B' S_B \to AB}\left(\rho_{A'B'} \otimes \theta_{S_A S_B}\right)$$

- In this case

$$Q^{2\to 2}_{PPT}(\mathcal{N}_{A'B'\to AB}) \leq E_R^{\mathrm{PPT}}(\theta_{S_A S_B}),$$
$$P^{2\to 2}_{LOCC}(\mathcal{N}_{A'B'\to AB}) \leq E_R(\theta_{S_A S_B}),$$

- Example: CNOT gate teleportation simulable, with θ it's Choi state.

Summary and Outlook

- Considered bipartite quantum interactions.
- Strong converse upper bound on PPT-assisted bidirectional quantum capacity.
- Efficiently computable by SDP.
- Strong converse upper bound on LOCC-assisted bidirectional private capacity.
- Application: Private reading.
- Other applications.
- Multipartite interactions.

Thank you for your attention!