Methods for the verification of bound entanglement

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arXiv:1804.07562
What is bound entanglement?

Distillable entanglement

A state $\rho_{AB}$ is distillable, if

- having some finite number of copies it is possible to create
- by means of LOCC (local operations and classical communication)

at least one maximally entangled state $|\phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$.

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Do all bound entangled states have a PPT?

Often implied: PPT entangled $\iff$ bound entangled.
A multipartite state is bound entangled if
• it is entangled,
• but undistillable for all bipartitions.

Example: Smolin state
\[ \rho_{ABCD} = \frac{1}{4} (\Phi^+ + \Phi^- + \Psi^+ + \Psi^-) \]

Properties:
• globally entangled
• separable with respect to all bipartitions

Feels like cheating.

Methods for the verification of bound entanglement, p. 4
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Experiments

Multipartite:
- Amselem & Bourennane, Nature Phys. (two.osf/zero.osf/zero.osf/nine.osf)
- Barreiro et al., Nature Phys. (two.osf/zero.osf/one.osf/zero.osf)
- Kampermann et al., PRA (two.osf/zero.osf/one.osf/zero.osf)

Bipartite:
- DiGuglielmo et al., PRL (two.osf/zero.osf/one.osf/one.osf)
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Question. What was the statistical significance in those experiments?
Nowhere specified.

Methods for the verification of bound entanglement, p. 5
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Certification of bound entanglement

Protocol in use.

1. Perform state tomography,
2. reconstruct state,
3. bootstrap, determine whether bound entangled,
4. report fraction of bootstrapped states with bound entanglement.

Sounds decent, yields utterly unreliable results.

• Theorem: There can be no reliable state reconstruction. [Schwemmer et al., PRL (zero.osf/one.osf/five.osf)]
• Bound entangled states are high-dimensional & nonconvex set.
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Proper statistical analysis

If $\rho_0$ admits a bound entangled ball with radius $r_0$, then we can compute an upper bound for $P[\text{data looks good} | \|\rho_0 - \rho_{\text{exp}}\|_2 \geq r_0]$

Advantages:
- easy to understand
- correct
- computationally trivial

Disadvantages:
- slightly conservative
- requires to work in Gaussian regime

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For a bound entangled state $\rho_0$, find $r_0$ such that all states $\tau$ with $\|\rho_0 - \tau\|_2 \leq r_0$ are bound entangled.
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Infeasible problem?
Task.

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.ImageIcon

Infeasible problem?

(We only consider the bipartite case.)
**Theorem (Horodecki)**

\[ \rho \text{ is undistillable if } \Gamma(\rho) \geq 0. \]
Theorem (Horodecki$^3$)

$\rho$ is undistillable if $\Gamma(\rho) \geq 0$.

Lemma. If $\|\rho_0 - \tau\|_2 \leq r_0$ then,

$$\lambda_{\min}[\Gamma(\tau)] \geq \lambda_{\min}[\Gamma(\rho_0)] - r_0 \sqrt{\frac{d-1}{d}}.$$
Theorem (Horodecki\textsuperscript{3})

\( \rho \) is undistillable if \( \Gamma(\rho) \geq 0 \).

**Lemma.** If \( \|\rho_0 - \tau\|_2 \leq r_0 \) then, \hspace{1cm} (\( d \): dimension of joint system)

\[
\lambda_{\min}[\Gamma(\tau)] \geq \lambda_{\min}[\Gamma(\rho_0)] - r_0 \sqrt{\frac{d-1}{d}}.
\]

**Corollary.**

All states around \( \rho_0 \) are undistillable, if

\[
\lambda_{\min}[\Gamma(\rho_0)] \geq r_0 \sqrt{\frac{d-1}{d}}.
\]
Computable cross-norm or realignment (CCNR) criterion:

Let $\left( g_k \right)_k$ be an orthonormal basis of the Hermitian operators and define $R(\rho)_{k,\ell} = \text{tr}(\rho g_k \otimes g_\ell)$.

Then, a state $\rho$ is entangled if $\|R(\rho)\|_1 > 1$.

Lemma:

If $\|\rho_0 - \tau\|_2 \leq r_0$, then $\|R(\tau)\|_1 \geq \|R(\rho_0)\|_1 - r_0 \sqrt{d}$.

Corollary.

All states around $\rho_0$ are entangled, if $\|R(\rho_0)\|_1 > 1 + r_0 \sqrt{d}$.
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**Lemma:** If \(\| \rho_0 - \tau \|_2 \leq r_0\), then
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Conditions

\begin{align*}
\text{C1: } & \lambda_{\min}[\Gamma(\tau)] \geq \lambda_{\min}[\Gamma(\rho_0)] - r_0 \sqrt{\frac{d-1}{d}}. \\
\text{C2: } & \|R(\tau)\|_1 \geq \|R(\rho_0)\|_1 - r_0 \sqrt{d}. 
\end{align*}
Optimal states

Clearly, we can find the best $r_0$.

Why not search a state $\rho_0$ with overall maximal $r_0$?

Optimization problem. Find $\rho$ and $r$ subject to

$$\text{maximize: } r$$
$$\text{such that: } \lambda_{\text{min}}[\Gamma(\rho)] \geq r \sqrt{d - 1}/d,$$
and
$$\|R(\rho)\|_1 > 1 + r \sqrt{d}.$$
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\end{align*}$$

- In principle, can be applied to given dimension.
- Practically, need to choose family of states with few parameters.
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- Practically, need to choose family of states with few parameters.
Example: Qutrits

Family of states: (contains Horodecki states)

\[
\rho = a |\phi_3 \rangle \langle \phi_3 | + b \sum_{k=0}^{2} |k, k \oplus 1 \rangle \langle k, k \oplus 1 | + c \sum_{k=0}^{2} |k, k \oplus 2 \rangle \langle k, k \oplus 2 | ,
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[Baumgartner et al., PRA (2006)]
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Can be solved analytically.

Optimal parameters

\[ a \approx 0.21289, \ b \approx 0.04834, \ \text{and} \ c \approx 0.21403. \]

\[ r_0 \approx 0.02345 \]

• Rank-7 state.

• Value of \( r_0 \) is (basically) tight w.r.t. CCNR and PPT.
Example: Ququarts

Bloch-diagonal states: (contain Smolin state)

\[ \rho = \sum_{k} x_k g_k \otimes g_k, \]

where \( g_k = (\sigma_\mu \otimes \sigma_\nu)/2. \)
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Feasibility polytope can be determined, has 254 556 vertices.
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Feasibility polytope can be determined, has 254,556 vertices.

**Optimal states**

- \( \text{rank}(\rho) < 9 \) yields \( r_0 = 0. \)
- \( \text{rank}(\rho) = 9 \) yields \( r_0 \approx 0.0161. \)
- \( \text{rank}(\rho) \geq 10 \) yields \( r_0 \approx 0.0214. \)
How large is 0.02?...some words about statistics

Protocol

/one.osf

Characterize tomography measurements with high precision.

two.osf

Decide critical statistical parameters.

three.osf

Perform state tomography.

/four.osf

Evaluate $\chi^2$-test.

two.osf

Publish or perish.

Statistical parameters:

• distribution of raw data (Poissonian, multinomial, ...)

• preprocessing method ($\text{raw data} \rightarrow x$)

• (Covariance matrix $\Sigma$ of $x$)

• Quadratic test function $\hat{t}: x \rightarrow t$

• Threshold significance, yielding critical value $t^* (r_0)$

Methods for the verification of bound entanglement, p. /five.osf
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Methods for the verification of bound entanglement, p. 15
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Methods for the verification of bound entanglement, p. 15
Choice of test function

A good choice of the test function is

\[ \hat{t}(x) = \| \Sigma^{-1/2} [T(\rho_0) - x] \|_2 \]

with \( T(\rho_0) \) the expected value of \( x \) for \( \rho_0 \).
Evaluation of the data

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with $T(\rho_0)$ the expected value of $x$ for $\rho_0$.

<arrow> Computable threshold value $t^*$, so that

$$P[\text{false positives}] \leq P[\hat{t}(x) \leq t^* \mid \|\rho_0 - \rho_{\text{exp}}\| > r_0] \leq \text{threshold significance}$$
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Certification of bound entanglement if \( \hat{t}(x) \leq t^* \).
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    & \leq \text{threshold significance}
\end{align*} \]

Certification of bound entanglement if \( \hat{t}(x) \leq t^* \).

Even with \( \|\rho_0 - \rho_{\text{exp}}\| \leq r_0 \), there is a chance that \( \hat{t}(x) > t^* \). These unlucky cases reduce with more samples.
Precision requirements

- Probability $p_{\text{fail}}$ to obtain data
  - which does not confirm bound entanglement
  - at a level of significance of $k\sigma$ standard deviations
  - assuming 5% (2.5%) white noise for qutrit (ququart) case.
Concluding remark

Preparation of bound entangled states requires the preparation of high-rank mixed states.
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Which strategy is admissible?

1. Prepare a purification of $\rho$ and discard auxiliary system.
2. Randomly prepare eigenstates of $\rho$.
3. Perform tomography of each of the eigenstates of $\rho$. 

Concluding remark

Preparation of bound entangled states requires the preparation of high-rank mixed states.

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2. Randomly prepare eigenstates of $\rho$.
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Sentís, Greiner, Shang, Siewert, K, arXiv:1804.07562