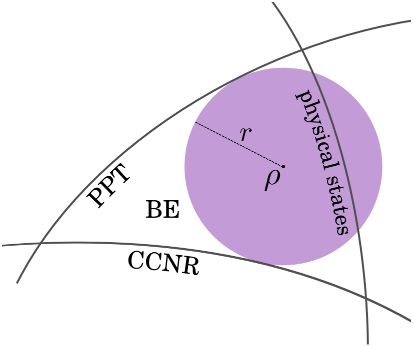
Methods for the verification of bound entanglement

Matthias Kleinmann, Universität Siegen
joint work with G. Sentís, J.N. Greiner, J. Shang, and J. Siewert



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Bound entanglement

An entangled state which is not distillable is bound entangled.

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Often implied: PPT entangled $\Longleftrightarrow$ bound entangled.

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Feels like cheating…

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1. Perform state tomography,
2. reconstruct state,
3. bootstrap, determine whether bound entangled,
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Problems

- Theorem: There can be no reliable state reconstruction.
  [Schwemmer *et al.*, PRL (2015)]
- Bound entangled states are high-dimensional & nonconvex set.

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Disadvantages:

- slightly conservative
- requires to work in Gaussian regime

Task.

For a bound entangled state ρ_0, find r_0 such that all states τ with $\|\rho_0 - \tau\|_2 \leq r_0$ are bound entangled.

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(We only consider the bipartite case.)

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$$\lambda_{\min}[\Gamma(\tau)] \geq \lambda_{\min}[\Gamma(\rho_0)] - r_0\sqrt{\tfrac{d-1}{d}}.$$

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Corollary.

All states around ρ_0 are undistillable, if

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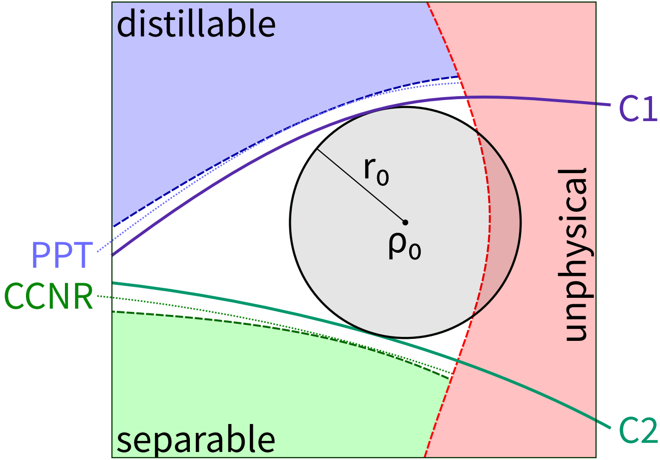
All states around ρ_0 are entangled, if

$$\|R(\rho_0)\|_1 > 1 + r_0\sqrt{d}.$$

Conditions



- C1: $\lambda_{\min}[\Gamma(\tau)] \geq \lambda_{\min}[\Gamma(\rho_0)] - r_0\sqrt{\frac{d-1}{d}}$.
- C2: $\|R(\tau)\|_1 \geq \|R(\rho_0)\|_1 - r_0\sqrt{d}$.

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Optimization problem.

Find ρ and r subject to

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- In principle, can be applied to given dimension.
- Practically, need to choose family of states with few parameters.

Example: Qutrits

Family of states: (contains Horodecki states)

$$\rho = a|\phi_3\rangle\langle\phi_3| + b\sum_{k=0}^{2}|k, k\oplus 1\rangle\langle k, k\oplus 1| + c\sum_{k=0}^{2}|k, k\oplus 2\rangle\langle k, k\oplus 2|\,,$$

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Optimal parameters

$a \approx$ 0.21289, $b \approx$ 0.04834, and $c \approx$ 0.21403.

↪ $r_0 \approx$ 0.02345

- Rank-7 state.
- Value of r_0 is (basically) tight w.r.t. CCNR and PPT.

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Optimal states

- $\mathrm{rank}(\rho) < 9$ yields $r_0 = 0$.
- $\mathrm{rank}(\rho) = 9$ yields $r_0 \approx 0.0161$.
- $\mathrm{rank}(\rho) \geq 10$ yields $r_0 \approx 0.0214$.

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Statistical parameters:

- distribution of raw data (Poissonian, multinomial, …)
- preprocessing method (raw data) $\mapsto \boldsymbol{x}$.
- (Covariance matrix Σ of x.)
- Quadratic test function $\hat{t}\colon \boldsymbol{x} \mapsto t$.
- Threshold significance, yielding critical value $t^*(r_0)$.

Evaluation of the data

Choice of test function

A good choice of the test function is

$$\hat{t}(\boldsymbol{x}) = \|\Sigma^{-1/2}[T(\rho_0) - \boldsymbol{x}]\|_2$$

with $T(\rho_0)$ the expected value of $\boldsymbol{x}$ for ρ_0.

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$\hookrightarrow$ Computable threshold value t^*, so that

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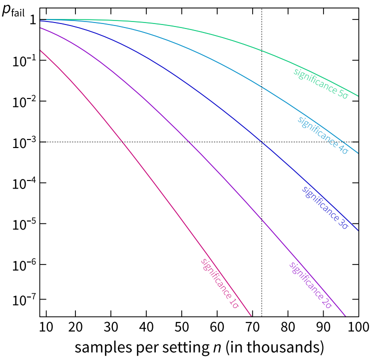
Even with $\|\rho_0 - \rho_{\text{exp}}\| \leq r_0$, there is a chance that $\hat{t}(\boldsymbol{x}) > t^*$.
These unlucky cases reduce with more samples.

Precision requirements

qutrit



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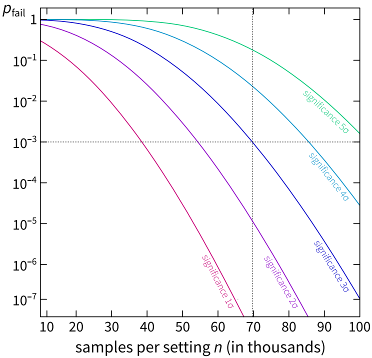

Probability p_{fail} to obtain data

- which does not confirm bound entanglement
- at a level of significance of $k\sigma$ standard deviations
- assuming 5% (2.5%) white noise for qutrit (ququart) case.

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