Coherifying quantum states and channels

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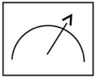
Outline

Table of contents

Motivating examples

Quantum state ρ

$\rho \longrightarrow$  $\longrightarrow p_j = \langle j|\rho|j\rangle$

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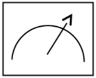
$\rho \longrightarrow$  $\longrightarrow p_j = \langle j|\rho|j\rangle$

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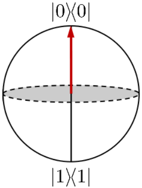

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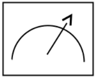
$\rho \longrightarrow$  $\longrightarrow p_j = \langle j|\rho|j\rangle$

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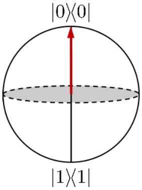

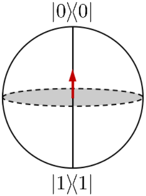

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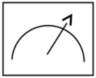
$\rho \longrightarrow$  $\longrightarrow p_j = \langle j|\rho|j\rangle$

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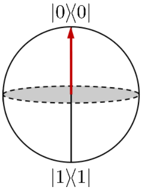

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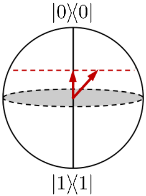

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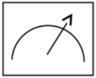
$\rho \longrightarrow$  $\longrightarrow p_j = \langle j|\rho|j\rangle$

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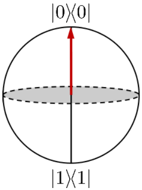

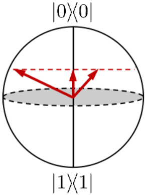

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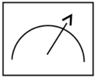
$\rho \longrightarrow$  $\longrightarrow p_j = \langle j|\rho|j\rangle$

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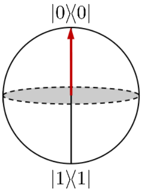

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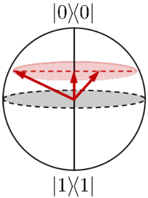

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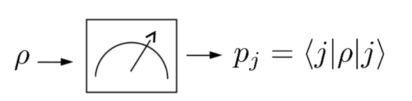

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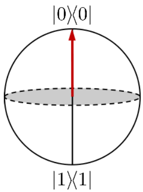

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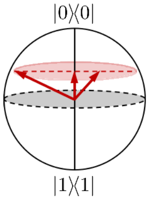
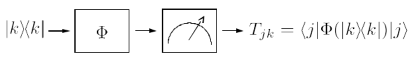


Quantum channel Φ



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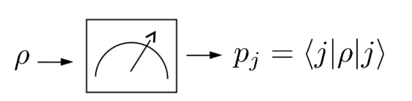

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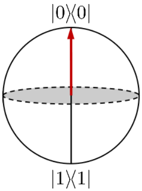

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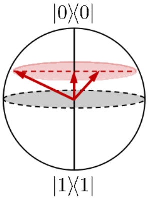

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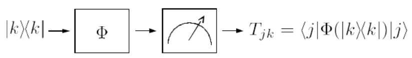
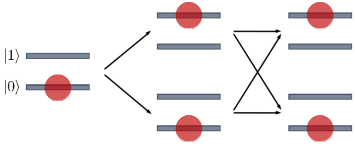


What T tells us about Φ?

$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \Phi(\cdot) = \frac{1}{2}\mathbf{1}$$



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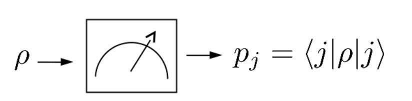

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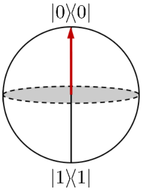

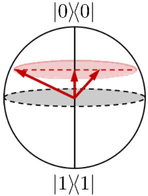

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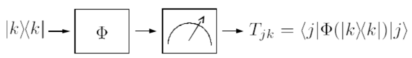

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$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \Phi(\cdot) = H(\cdot)H^\dagger, H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

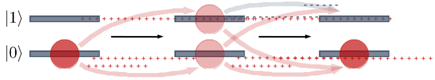

Table of contents

Coherence of quantum states

Given a fixed basis $\{j\}$ with $j \in \{1, 2, \ldots, d\}$
$\langle j|\rho|j\rangle$: occupations p_j
$\langle j|\rho|k\rangle$: coherences c_{jk}

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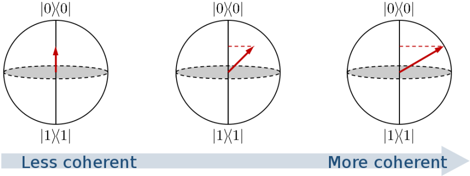

$$\rho_1 = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \qquad \rho_2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad \rho_3 = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

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$$\mathcal{D}(\rho) = \sum_j \langle j|\rho|j\rangle |j\rangle\langle j|$$

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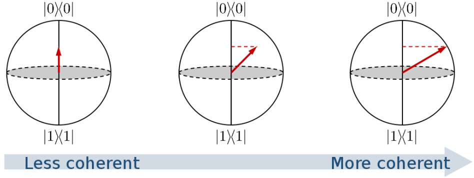

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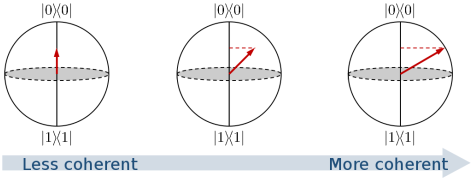

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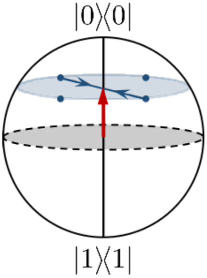

$$c_{jk} \to 0, p_j \to p_j$$

$$\mathcal{D}(\rho_1) = \mathcal{D}(\rho_2) = \mathcal{D}(\rho_3) = \rho_1.$$

Coherence of quantum states

Incoherent (classical) state ρ is identified with probability distribution p.

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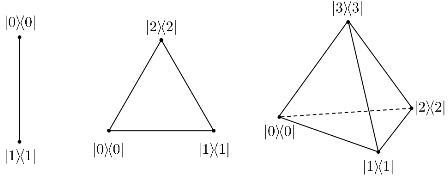

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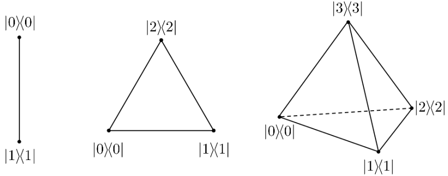

Coherence measures (distance from incoherent states):

$$C_e(\rho) = S(\rho || \mathcal{D}(\rho)) = S(p) - S(\lambda(\rho))$$
$$C_2(\rho) = \|\rho - \mathcal{D}(\rho)\|_{HS}^2 = \lambda(\rho) \cdot \lambda(\rho) - p \cdot p$$

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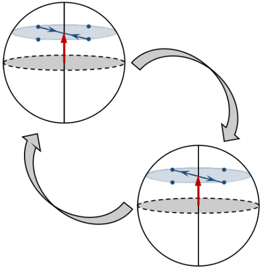

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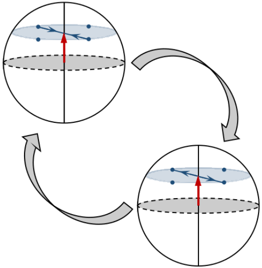

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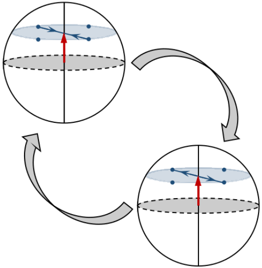

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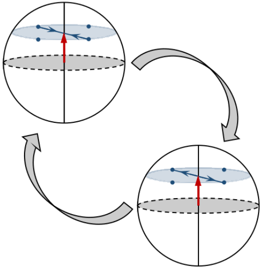

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How many distinct ways to coheryfy?

Coherence of quantum channels

Given a fixed basis $\{|j\rangle\}$,
with $j \in \{1, 2, \ldots, d\}$

$\langle j|\Phi(|k\rangle\langle k|)|j\rangle$: classical action T_{jk}
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Choi-Jamiołkowski isomorphism
channel $\Phi \longleftrightarrow$ bipartite state J_Φ

$J_\phi = \frac{1}{d}(\Phi \otimes \mathbb{1})|\Omega\rangle\langle\Omega|$, $|\Omega\rangle = \sum_j |jj\rangle$

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Vectorising classical action:

$diag(J_\Phi) = \frac{1}{d}|T\rangle\rangle$, where $|T\rangle\rangle = T \otimes \mathbb{1}|\Omega\rangle$

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Define a coherence measures of Φ through coherence measures of J_Φ

$$C_e(\Phi) = S(\frac{1}{d}|T\rangle\rangle) - S(\lambda(J_\Phi)), \quad C_2(\Phi) = \lambda(J_\Phi) \cdot \lambda(J_\Phi) - \frac{1}{d^2}\langle\langle T||T\rangle\rangle$$

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$$C_e(\Phi) = S(\frac{1}{d}|T\rangle\rangle) - S(\lambda(J_\Phi)), \quad C_2(\Phi) = \lambda(J_\Phi)\cdot\lambda(J_\Phi) - \frac{1}{d^2}\langle\langle T||T\rangle\rangle$$

Similarly to:

$$C_e(\rho) = S(\rho||\mathcal{D}(\rho)) = S(p) - S(\lambda(\rho))$$
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Coherence of quantum channels

Classical channels are defined as channels with incoherent (classical) Jamiołkowski state.

Action of classical channel with classical action T

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Optimising coherence of Φ with fixed $T \iff$ optimising $\lambda(J_\Phi)$.

Coherence of quantum channels

Decohering operation $\mathcal{D}$

Φ with $diag(J_\Phi) = \frac{1}{d}|T\rangle\rangle \mapsto \Phi^{\mathcal{D}}$ with $J_{\Phi^{\mathcal{D}}} = \mathcal{D}(J_\Phi) = \frac{1}{d} diag(|T\rangle\rangle)$

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Example

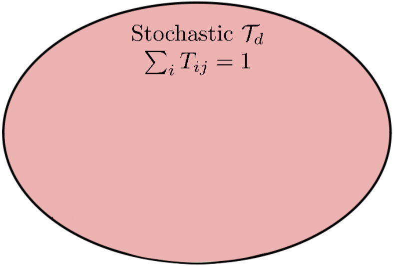
$$T = \left[\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right]$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\mathrm{tr}_1|\psi\rangle\langle\psi| = |+\rangle\langle+|$$

Table of contents

Categories of classical actions



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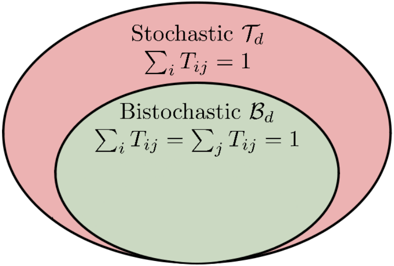

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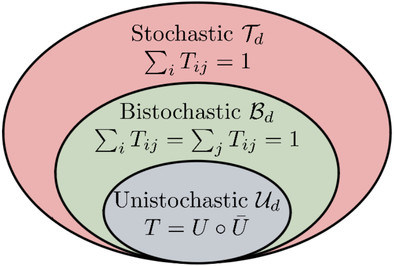

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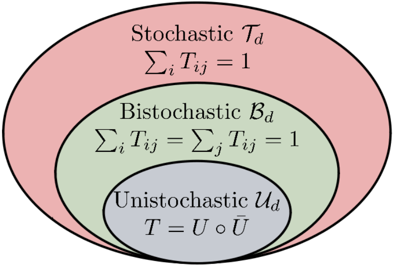

Hadamard product:
$(A \circ B)_{jk} = A_{jk} B_{jk}$

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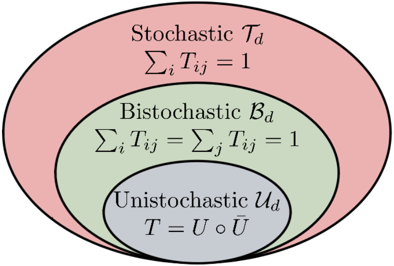

Example of $T \in B_d$ and $T \notin \mathcal{U}_d$

$$T = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad U = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & e^{i\theta_{12}} & e^{i\theta_{13}} \\ e^{i\theta_{21}} & 0 & e^{i\theta_{23}} \\ e^{i\theta_{31}} & e^{i\theta_{32}} & 0 \end{bmatrix},$$

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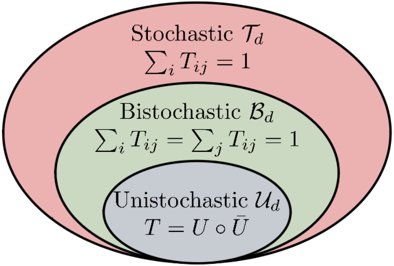

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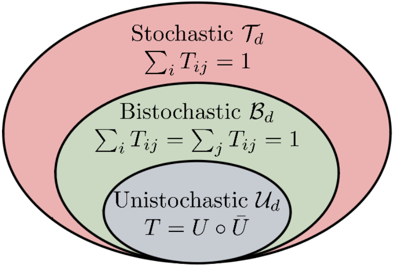

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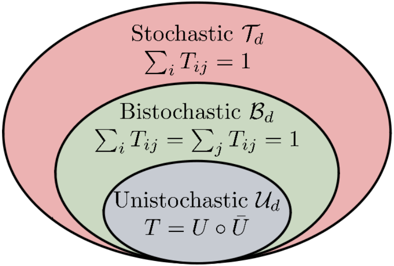

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Φ can be completely coherified $\iff$ T is unistochastic

Coherification upper-bound

Majorization partial order: $p \succ q \iff \forall_k \sum_{j=1}^k p_j^{\downarrow} \geq \sum_{j=1}^k q_j^{\downarrow}$

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Optimal coherification of qubit channels

Classical action of a qubit channel:

$$T = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} =: \begin{bmatrix} a & \tilde{b} \\ \tilde{a} & b \end{bmatrix}$$

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with unitary

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and $\Psi(\cdot) = L_1(\cdot)L_1^{\dagger} + L_2(\cdot)L_2^{\dagger}$ with

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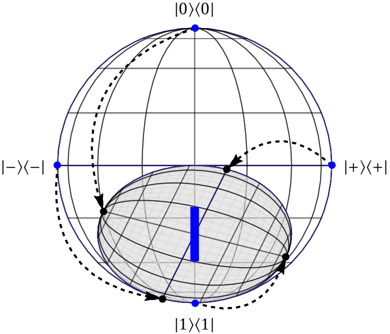

$$T = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

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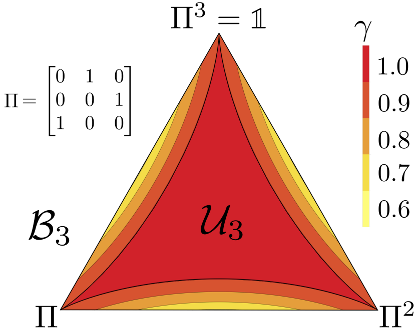

Developed a family of bounds based on polygon constraints

Coherence and classical randomness

Evolution of a pure quantum state $|\psi\rangle$ under the action of channel Φ

$$\Phi(|\psi\rangle\langle\psi|) = \sum_j K_j|\psi\rangle\langle\psi|K_j^\dagger$$

can be interpreted as incoherent mixture of pure state transformations

$$|\psi\rangle \mapsto \frac{1}{\sqrt{q_j}} K_j|\psi\rangle \quad \text{with probability } q_j = \mathrm{tr} K_j|\psi\rangle\langle\psi|K_j^\dagger.$$

Path probability averaged over all pure states,

$$\langle q_j\rangle_\psi = \mathrm{tr} K_j \langle|\psi\rangle\langle\psi|\rangle_\psi K_j^\dagger = \frac{1}{d}\mathrm{tr} K_j K_j^\dagger = \lambda_j(J_\Phi)$$

Table of contents

Perfectly distinguishable state coherifications

One can always optimally coherify state p

$$\rho = diag(p) \overset{\mathcal{C}}{\longmapsto} |\psi_j\rangle\langle\psi_j| \quad \text{with} \quad \psi = \sum_k \sqrt{p_k} e^{i\phi_{jk}} |k\rangle$$

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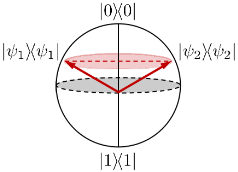

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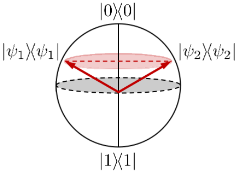

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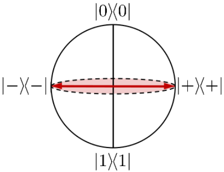

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Necessary condition for N-distinguishability

Necessary condition

N perfectly distinguishable states $\{\psi_i\}$ with $|\langle k|\psi_j\rangle|^2 = p_k$ $\Longrightarrow$ $\forall k : p_k \leq \frac{1}{N}$

Orthogonal $\{|\psi_j\rangle\}$ could form columns of unitary matrix

$$U = \begin{bmatrix} \sqrt{p_1} e^{i\phi_{11}} & \dots & \sqrt{p_1} e^{i\phi_{1N}} & \dots \\ \sqrt{p_2} e^{i\phi_{21}} & \dots & \sqrt{p_1} e^{i\phi_{2N}} & \dots \\ \vdots & \ddots & \vdots & \ddots \\ \sqrt{p_d} e^{i\phi_{d1}} & \dots & \sqrt{p_d} e^{i\phi_{dN}} & \dots \end{bmatrix}$$

Corresponding unistochastic matrix:

$$U \circ \overline{U} = \begin{bmatrix} p_1 & \dots & p_1 & \dots \\ p_2 & \dots & p_2 & \dots \\ \vdots & \ddots & \vdots & \ddots \\ p_d & \dots & p_d & \dots \end{bmatrix}$$

But rows must sum to 1!

Distinguishing channel coherifications

Channels $\{\Phi^{(j)}\}$ with fixed action T are perfectly distinguishable iff:

$\exists \rho_{AB}$ $\{\Phi^{(j)} \otimes \mathbb{1}(\rho_{AB})\}$ are perfectly distinguishable

If $\exists \rho$ $\{\Phi^{(j)}(\rho)\}$ are perfectly distinguishable then no entanglement is needed

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Type of classical action	Number of perfectly distinguishable channels	Requires entanglement
Unistochastic	d	No
Unistochastic	$d+1, \ldots, d^2$	Yes
Bistochastic	2	Yes
Such that $T_{jk} \leq \frac{1}{2}$	2	No

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