Useful correlations from bound entangled states

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CEQIP, June 2018

Abstract

- Bound entangled states are very weakly entangled states. In fact they are so weakly entangled that given an infinite number of copies, no pure state entanglement can be distilled from them. Nevertheless, they are useful in certain applications such as quantum key distribution. Here we show that bipartite bound entangled states are also useful in metrology and Bell nonlocality. In particular they can overcome the classical limit in quantum metrology and can give rise to Bell inequality violation.

Outline

1. Motivation
2. PPT bound entangled states
3. Bell nonlocality
3. Quantum metrology
5. Usefulness of PPT states

Motivation

In which applications are entangled states useful for?

Especially: What are very weakly entangled states useful for?

We focus on two areas:

Bell nonlocality and quantum metrology.

Motivation

Bell nonlocality:

Entanglement is required to exceed the local classical limit.

But not all entangled states are nonlocal. E.g. two-qubit Werner states:

$$\rho_W(p) = p\left|\Psi_-\right\rangle\left\langle\Psi_-\right| + (1-p)\frac{\mathrm{I}}{2}\otimes\frac{\mathrm{I}}{2}$$

They are entangled for p>1/3, but local for p<0.45 (for any POVM measurements).

Motivation

Quantum metrology:

Entanglement is required to overcome the classical limit.

But not all entangled states are useful metrologically. Some of the cluster states are not useful: e.g. for more than 4 particles, ring cluster states, as well as more than one dimensional cluster states (Hyllus, Gühne, Smerzi 2010).

Motivation

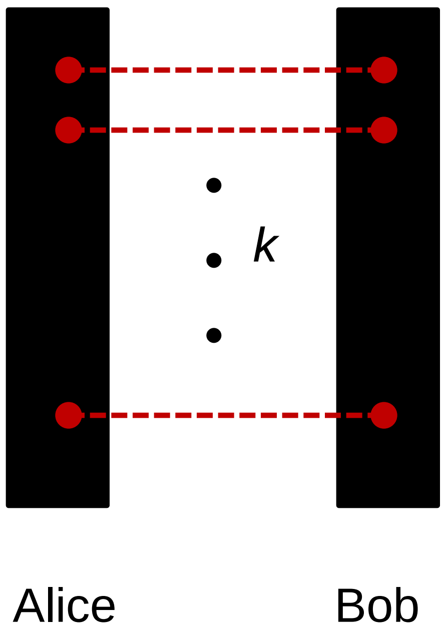
We have seen that relatively highly entangled states can be local or useless in metrology.

But what can we say about the weakly entangled bound entangled states?

First we describe a class of bound entangled states, so-called PPT states and then show that some of them are useful both in Bell nonlocality and metrology.

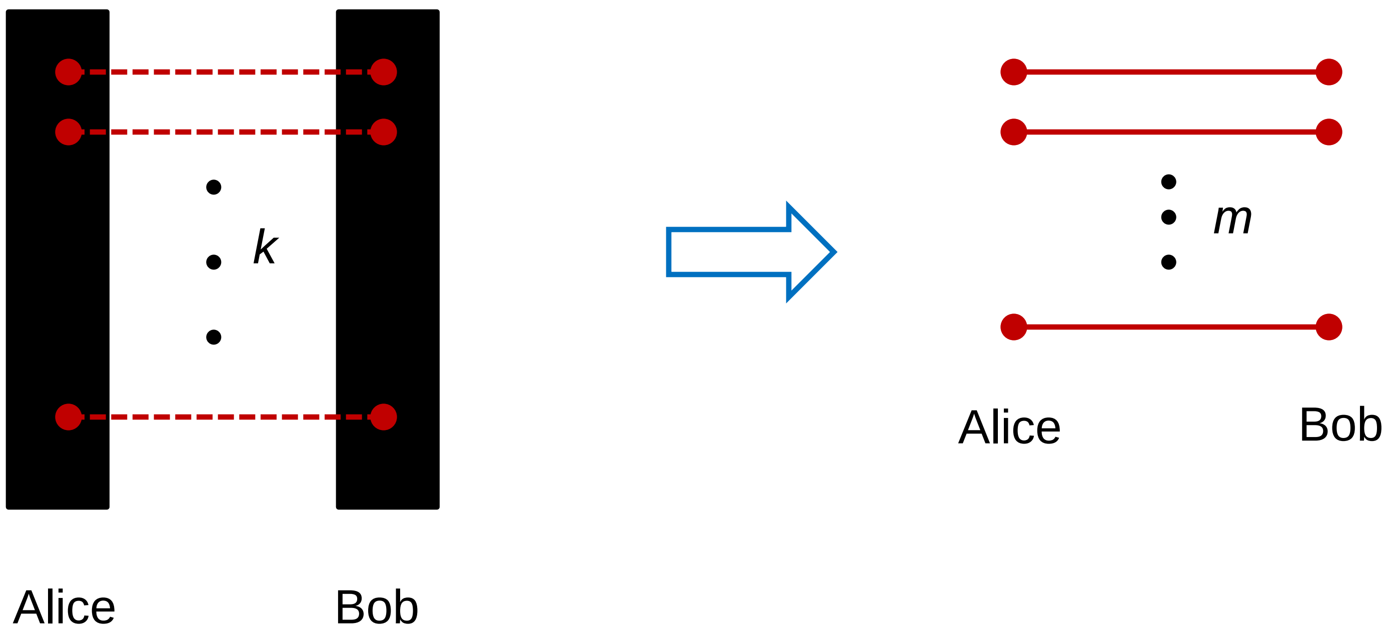
Bound entanglement

Suppose that Alice and Bob share *k* copies of a mixed state ρ_{AB}:



Bound entanglement

Suppose that Alice and Bob share *k* copies of a mixed state ρ_{AB}: They can run a distillation protocol (LOCC) to extract singlet pairs:



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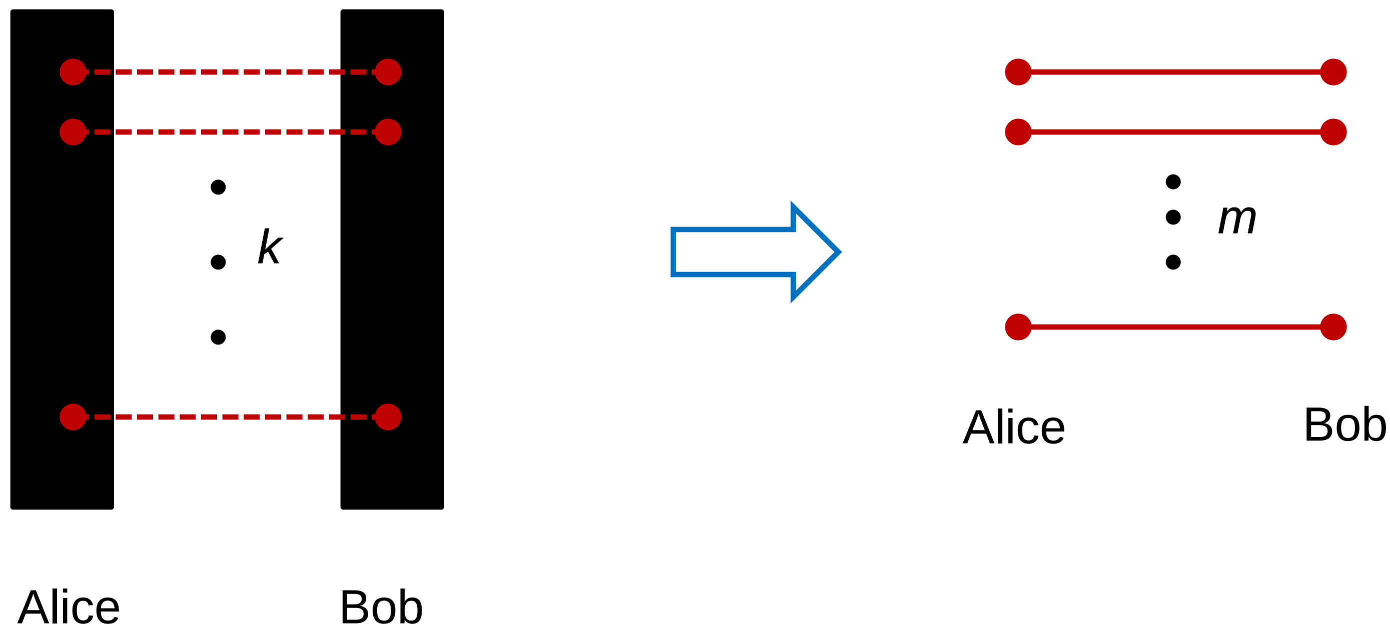

As a result they end up with *m*<*k* copies of a singlet state, which can be used for quantum information purposes.

Bound entanglement

Every two-qubit entangled state ρ_{AB} can be used to distill singlets with the above distillation protocol. Is it also true for systems of higher dimension?

Horodeckis proved in 1998 that this is not the case, and that there exist noisy entangled states in higher dimensions that cannot be distilled by local operations and classical operations (LOCC) into the singlet state.

These states are called bound entangled states. The smallest example provided by Horodeckis is a 3x3 dimensional state.

Bound entanglement

Given a state ρ_{AB} . How to decide if it is undistillable?
It is a difficult question in general, since there is no restriction on the specific type of LOCC operations or on the number of copies used in the distillation protocol.

Bound entanglement

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It is a difficult question in general, since there is no restriction on the specific type of LOCC operations or on the number of copies used in the distillation protocol.

Still there is a sufficient condition to undistillability:

If a state is positive under the partial transposition map (the state is so-called PPT), then the state is undistillable.

PT map: $\mathrm{PT}(\rho_{AB}) = (\mathrm{I} \otimes T_B)(\rho_{AB})$

Such states, provided they are entangled, are called PPT bound entangled states.

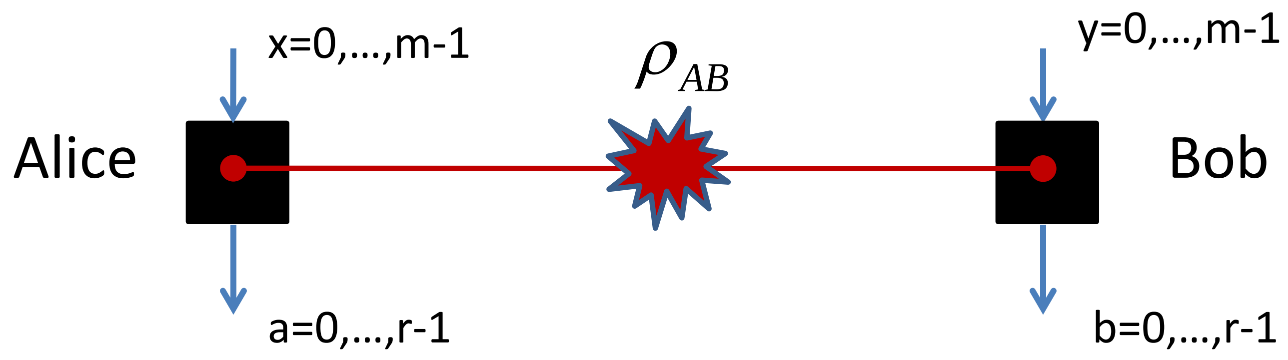
Bell nonlocality

- Are bound entangled states useful in creating non-local correlations, i.e. ones which cannot be simulated by classical resources? In particular, is it possible to violate Bell inequalities using bound entangled states?

- This is what we explore next. First we discuss a standard Bell nonlocality setup.

Bell nonlocality

Bell scenario: distant parties (Alice and Bob) choose between *m* different measurements of *r* outcomes.



Experimental data: P(a,b|x,y)

J.S. Bell: On the einstein-podolsky-rosen paradox, 1964

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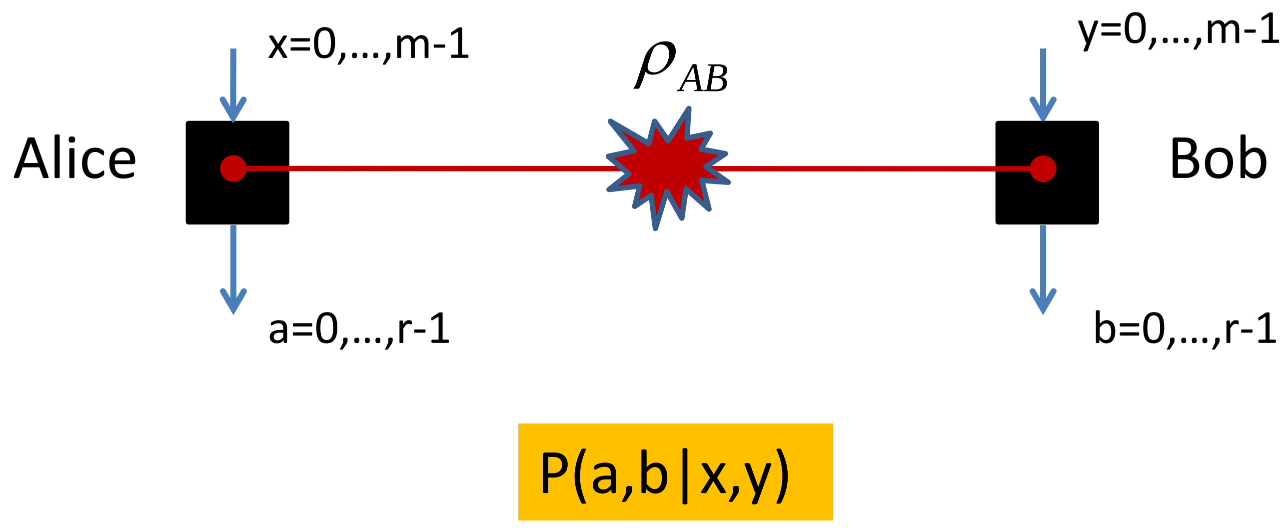
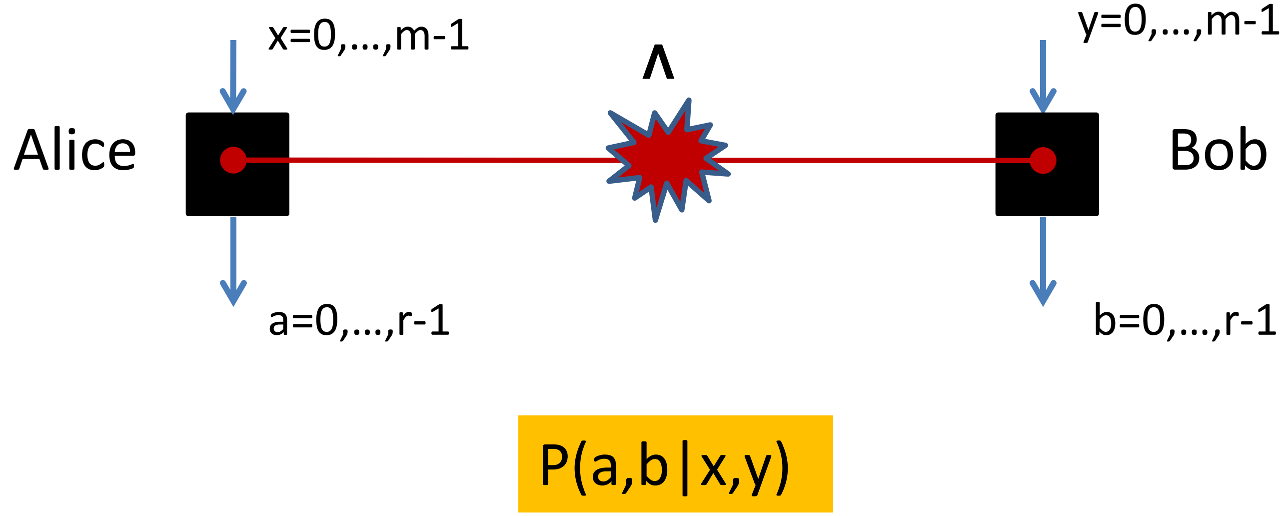


The set of quantum correlations is defined by

$$P(a,b\,|\,x,y) = \mathrm{tr}\left(\rho_{AB} M_{a|x} \otimes M_{b|y}\right)$$

Bell nonlocality

Bell scenario: distant parties (Alice and Bob) choose between *m* different measurements of *r* outcomes. **Λ** defines a classical, random source.



The set of local correlations is defined by

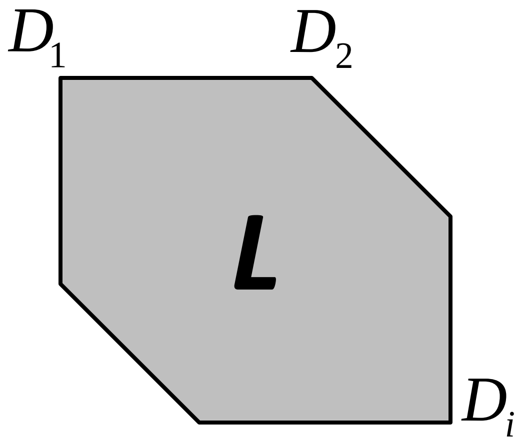
$$P(a,b\,|\,x,y)=\sum_{\lambda}p_{\lambda}P(a\,|\,x,\lambda)P(b\,|\,y,\lambda)$$

Bell nonlocality

$$P(a,b|x,y)=\sum_{\lambda} p_{\lambda} P(a|x,\lambda)P(b|y,\lambda)$$

Geometrically, the set ***L*** of local distributions is the convex hull of a finite number of points: it is a polytope.

The vertices D_i correspond to deterministic strategies, In which case p_{λ} are 0 or 1.



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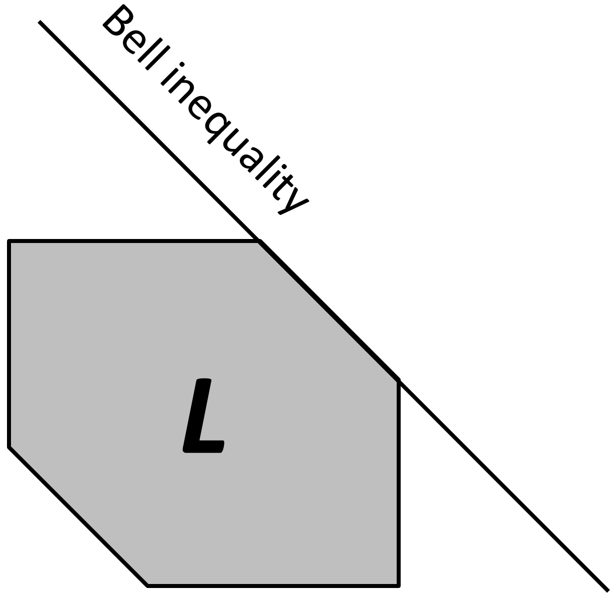
Bell inequalities define the limits on these local correlations.



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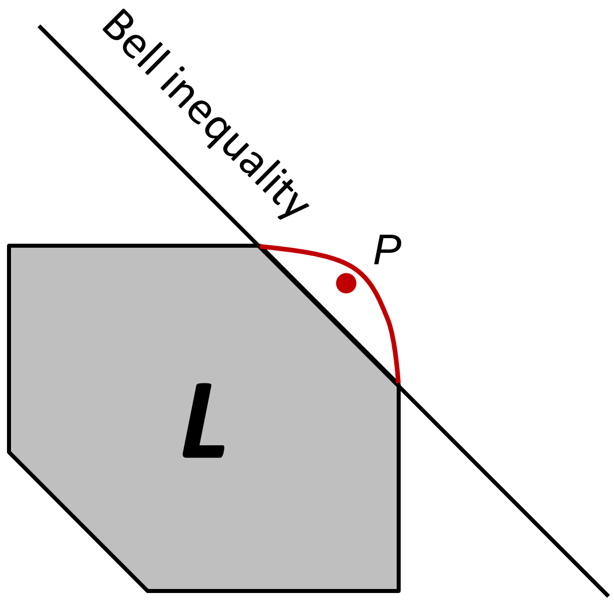
$$P(a,b\,|\,x,y)=\sum_{\lambda} p_{\lambda} P\left(a\,|\,x,\lambda\right) P\left(b\,|\,y,\lambda\right)$$

Geometrically, the set ***L*** of local distributions is the convex hull of a finite number of points: it is a polytope.

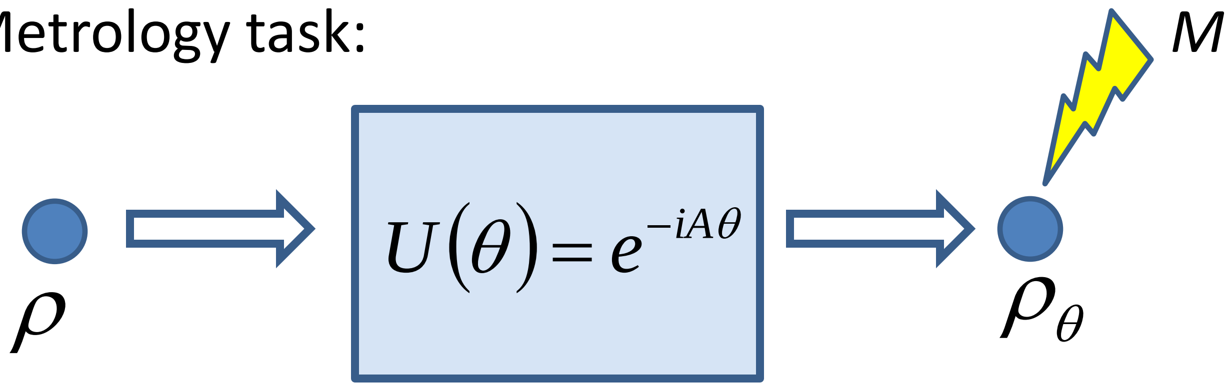
Every point outside the set ***L*** is called nonlocal.

Bell inequality is a very useful tool to detect nonlocal correlations. In the figure, *P* is detected to be nonlocal.



Metrology

- Metrology task:



The basic task is to estimate the parameter θ in the dynamics. In order to measure θ we prepare a probe state ρ, let it evolve, and finally measure the evolved state with an operator M.

ρ is said to be useful if it gives better performance than any separable state.

Metrology

The precision of the estimation of θ is given by the formula:

$$\frac{1}{(\Delta\theta)^2} = \frac{\left|\partial_\theta \langle M \rangle\right|^2}{(\Delta M)^2}$$

Accordingly, the precision of the estimation (left-hand side) depends on two things:

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Accordingly, the precision of the estimation (left-hand side) depends on two things:

- the sensitivity of the expectation value of *M* to the change of angle θ: the higher the sensitivity, the higher the precision.
- the variance of *M*: the larger the variance, the lower the precision.

Metrology

The precision of parameter θ is limited by the Cramér-Rao bound as

$$\frac{1}{(\Delta\theta)^2} \le F_Q(\rho, A)$$

where $F_Q(\rho, A)$ is the quantum Fisher information:

$$F_Q(\rho, A) = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k - \lambda_l} \left|\langle k \,|\, A \,|\, l \rangle\right|^2$$

where $\rho = \sum_k \lambda_k |k\rangle\langle k|$.

In linear interferometers *A* are collective operators. They are defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)} \qquad l \in \{x, y, z\}$$

Metrology

Metrology is linked to the entanglement of ρ as follows:

Shot noise limit: $F_Q(\rho, J_l) \leq N \qquad l = x, y, z$

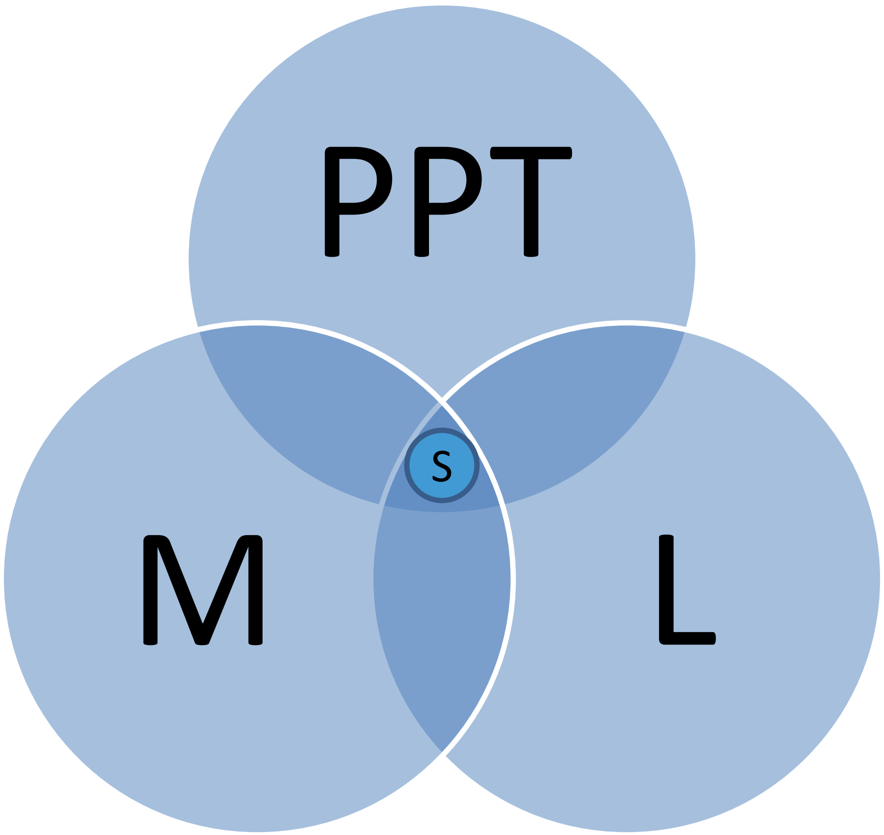
The above inequality holds for *N*-qubit separable states. A quantum state is useful in metrology if it violates the above inequality.

Heisenberg limit: $F_Q(\rho, J_l) \leq N^2 \qquad l = x, y, z$

where the inequality can be saturated (e.g. with GHZ or Dicke states).

Useful PPT states

Sets of metrologically useless states (*M*), PPT states (*PPT*), Bell local states (*L*). Separable states (*S*) are in the intersection of these sets.



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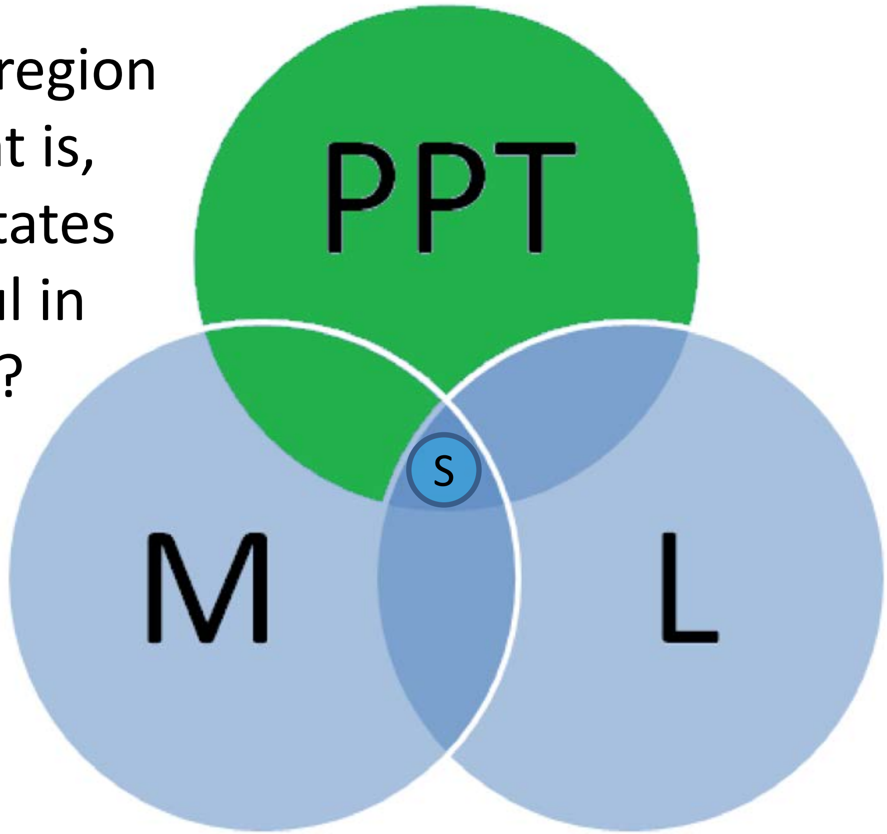
Q: Is the green region nonempty? That is, are there PPT states which are useful in Bell nonlocality?



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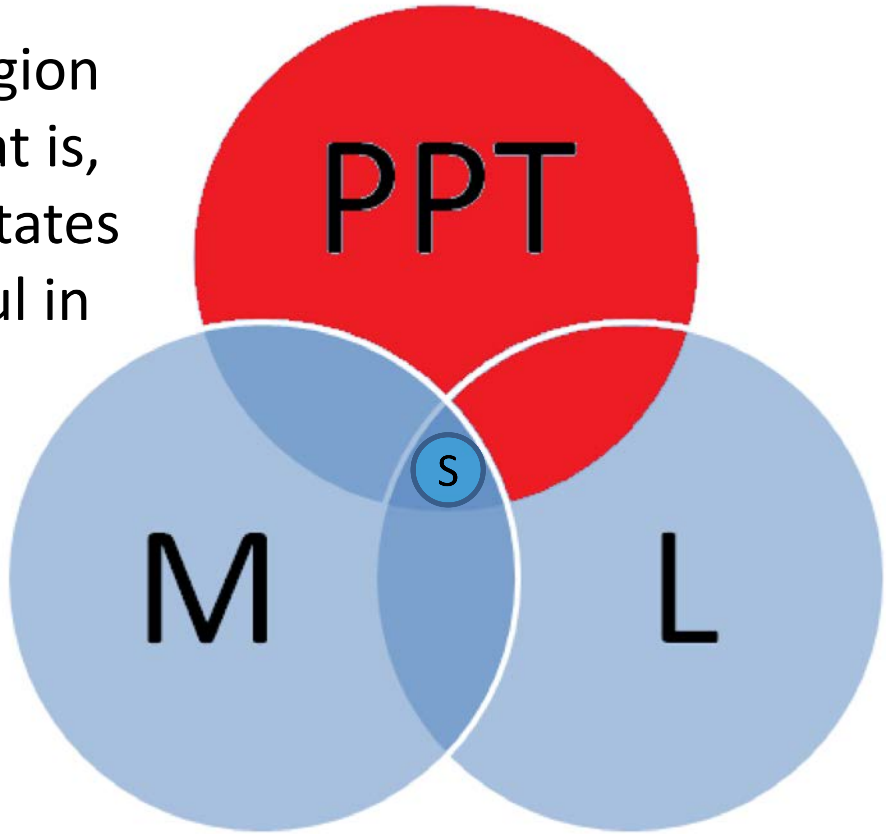

Useful PPT states in nonlocality

Bell nonlocality:
T. Vértesi, & N. Brunner (2014). Disproving the Peres conjecture by showing Bell nonlocality from bound entanglement. *Nature communications*, *5*, 5297.
An example of a bipartite state is presented within the **green region**.
Especially: a 3x3 dimensional PPT bound entangled state which violates a bipartite Bell inequality. It refutes Peres' conjecture.

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Especially: a 3x3 dimensional PPT bound entangled state which violates a bipartite Bell inequality. It refutes Peres' conjecture.

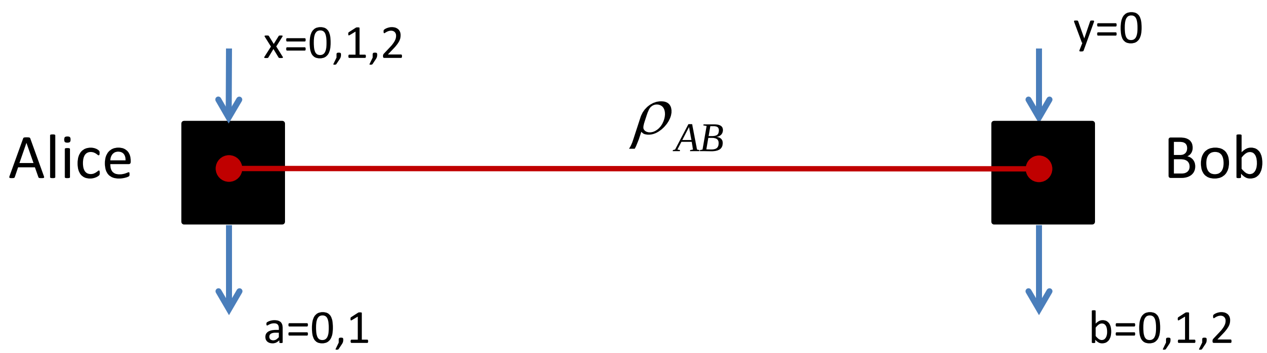
The state is of rank-4: $$\rho_{AB} = \sum_{i=1}^{4} \lambda_i |\psi_i\rangle\langle\psi_i|$$

It fulfills PT invariance: $$\mathrm{PT}(\rho_{AB}) = (\mathrm{I} \otimes T_B)\rho_{AB} = \rho_{AB}$$

This ensures that the state is PPT and therefore undistillable. The PPT state is a miminal construction in terms of dimensions, since no PPT entangled state exists in dimensions 2x3 or 2x2.

Useful PPT states in nonlocality

Bell inequality (S. Pironio, 2014):



P(a,b|x,y)

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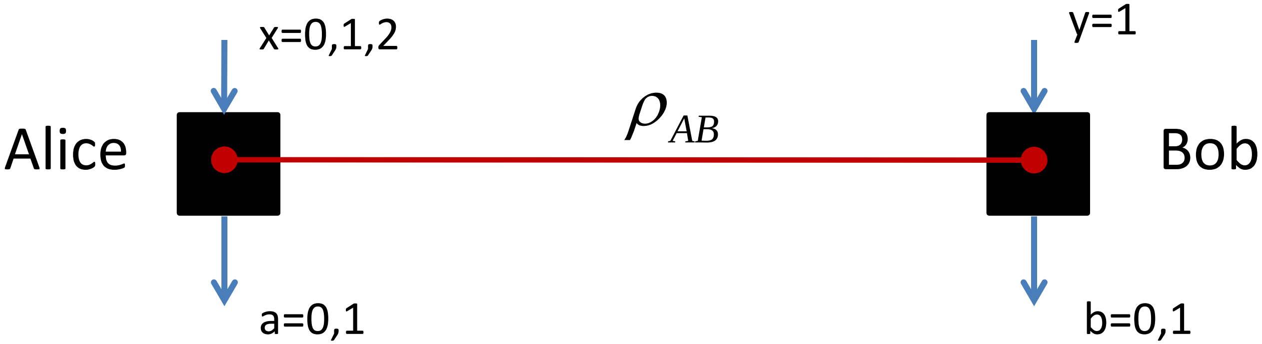

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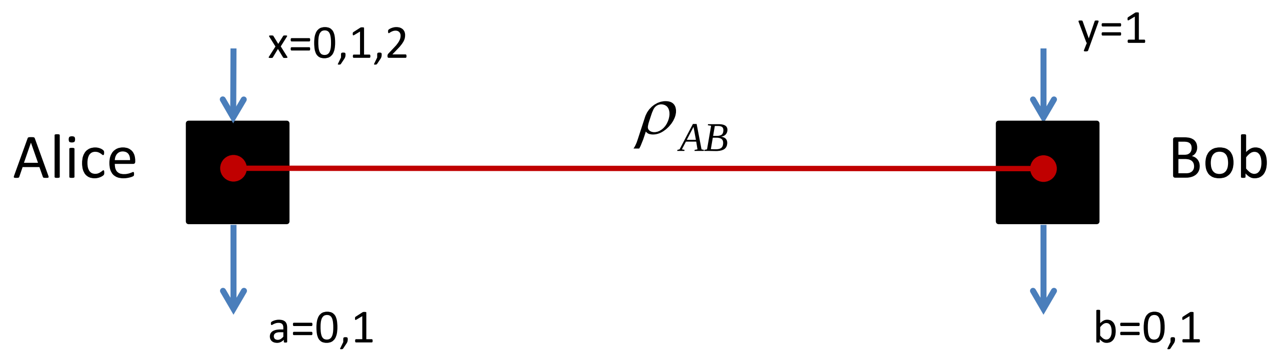


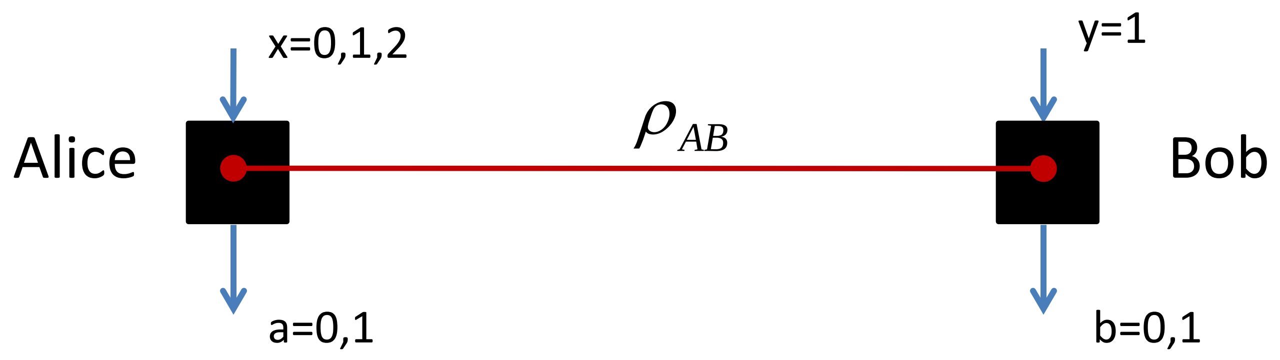
P(a,b|x,y) → Bell expression:

$$
\begin{aligned}
I = & -p_A(0|2) - 2p_B(0|1) - p(01|00) - p(00|10) + p(00|20) \\
& + p(01|20) + p(00|01) + p(00|11) + p(00|21)
\end{aligned}
$$

I <= 0 holds for all local P(a,b|x,y) distributions

Useful PPT states in nonlocality

Bell inequality (S. Pironio, 2014):



Quantumly: with the use of PPT state above

$$P(a,b|x,y) = \mathrm{tr}\left(\rho_{\mathrm{PPT}} M_{a|x} \otimes M_{b|y}\right)$$

we get larger than zero for the Bell value:

$$I_{\mathrm{PPT}} = \frac{-3386 + 18\sqrt{42} - 5\sqrt{131} + 45\sqrt{5502}}{43025} \simeq 2.63144 \times 10^{-4}$$

Useful PPT states in nonlocality

The construction above with $I_{\mathrm{PPT}} \cong 2.63144\times10^{-4}$
is analytical (both the states and measurements can be given in a closed form).

Using the SDP technique of Moroder et al. (PRL, 2013), one finds the following upper bound:

$$I_{\mathrm{PPT}}^{\max} \cong 4.8012\times10^{-4}$$

This leaves possible room for slightly bigger violation with PPT states in 3x3 or in higher dimensions.

Useful PPT states in nonlocality

How did we find the counterexample?

Let us consider a generic bipartite Bell inequality with local bound *L*:

$$I = \sum_{a,b,x,y} c_{a,b,x,y} P(a,b \mid x,y) \leq L$$

Our task is to maximize:

$$I_{\mathrm{PPT}} = \sum_{a,b,x,y} c_{a,b,x,y} \,\mathrm{tr}\left(\rho_{\mathrm{PPT}} M_{a|x} \otimes M_{b|y}\right)$$

among *d* x *d* PPT states ρ_{PPT} and *d*-dimensional measurement operators $M_{a|x}$ and $M_{b|y}$.

We use a heuristic search, the so-called see-saw method.

Useful PPT states in nonlocality

See-saw procedure to maximize: $I_{\mathrm{PPT}} = \sum_{a,b,x,y} c_{a,b,x,y} \mathrm{tr}\left(\rho_{\mathrm{PPT}} M_{a|x} \otimes M_{b|y}\right)$

Useful PPT states in nonlocality

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step 0: pick random $\left(M_{a|x}, M_{b|y}\right)$

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step 0: pick random $\left(M_{a|x}, M_{b|y}\right)$

step 1: $\left(M_{a|x}, M_{b|y}\right) \to \rho_{\mathrm{PPT}}$

Useful PPT states in nonlocality

See-saw procedure to maximize: $I_{\text{PPT}} = \sum_{a,b,x,y} c_{a,b,x,y} \operatorname{tr}\left(\rho_{\text{PPT}} M_{a|x} \otimes M_{b|y}\right)$

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step 1: $\left(M_{a|x}, M_{b|y}\right) \rightarrow \rho_{\text{PPT}}$

step 2: $\left(M_{a|x}, \rho_{\text{PPT}}\right) \rightarrow M_{b|y}$

Useful PPT states in nonlocality

See-saw procedure to maximize: $I_{\mathrm{PPT}} = \sum_{a,b,x,y} c_{a,b,x,y} \mathrm{tr}\left(\rho_{\mathrm{PPT}} M_{a|x} \otimes M_{b|y}\right)$

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step 3: $\left(M_{b|y}, \rho_{\mathrm{PPT}}\right) \rightarrow M_{a|x}$

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step 3: $\left(M_{b|y}, \rho_{\mathrm{PPT}}\right) \rightarrow M_{a|x}$

step 4: back to step 1

Useful PPT states in nonlocality

Our analytical state disproves Peres conjecture.

Peres Conjecture: Undistillable states admit a local model
(A. Peres, Foundations of Physics 29, 589-614 (1999)):

"Note that there exist inseparable quantum states that cannot be distilled into singlets. In particular, quantum states whose partial transpose has no negative eigenvalue have that property. Thus, if the preceding conjectures are correct, it follows that these peculiar inseparable quantum states violate no Bell inequality, and therefore, owing to Farkas's lemma, their statistical properties are compatible with the existence of local objective variables."

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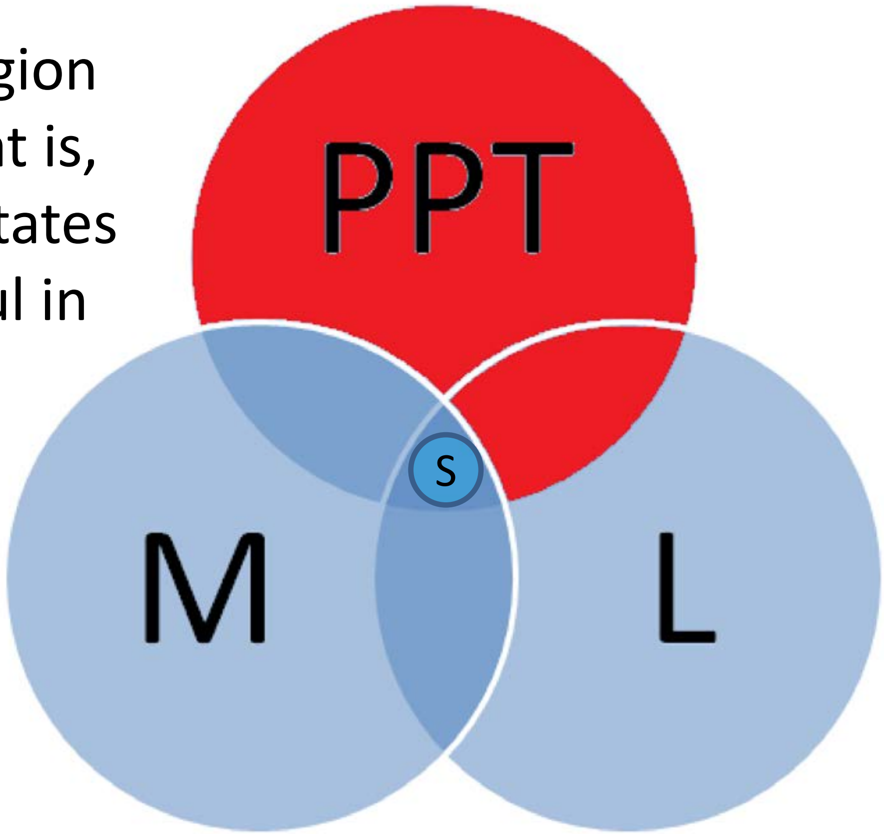

Useful PPT states in metrology

Results: G. Tóth, & T. Vértesi (2018). Quantum states with a positive partial transpose are useful for metrology. *Physical Review Letters*, *120*, 020506.

We find the following fully PPT states:

System	A	FQ	FQ_sep	p_white
four qubits	Jz	4.0088	4	0.0011
three qubits	jz(1) + jz(2)	2.0021	2	0.0005
2 x 4	jz(1) + jz(2)	2.0033	2	0.0008

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We find the following fully PPT states:

System	FQ	FQ_sep	p_white
3 x 3	8.0085	8	0.0003
4 x 4	9.3726	8	0.0382
12 x 12	11.3618	8	0.0808

Here *A* is not the usual *Jz* operator. It has the form:

$A = H \otimes I + I \otimes H$, where

$H = \mathrm{diag}(1,1,\ldots,\text{-}1,\text{-}1,\ldots)$ with the same number of +1's and -1's for even *d.*

Useful PPT states in metrology

In case of the 4x4 system, the bound entangled PPT state looks as follows. First we define the six states below:

$$|\psi_1\rangle = \frac{|0,1\rangle + |2,3\rangle}{\sqrt{2}} \qquad |\psi_2\rangle = \frac{|1,0\rangle + |3,2\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \frac{|1,1\rangle + |2,2\rangle}{\sqrt{2}} \qquad |\psi_4\rangle = \frac{|0,0\rangle - |3,3\rangle}{\sqrt{2}}$$

$$|\psi_5\rangle = \frac{|0,3\rangle + |1,2\rangle}{2} + \frac{|2,1\rangle}{\sqrt{2}}$$

$$|\psi_6\rangle = \frac{-|0,3\rangle + |1,2\rangle}{2} + \frac{|3,0\rangle}{\sqrt{2}}$$

Useful PPT states in metrology

Our PPT state is a convex mixture of the above defined states:

$$\rho_{\mathrm{PPT}} = p\sum_{n=1}^{4} |\psi_n\rangle\langle\psi_n| + q\sum_{n=5}^{6} |\psi_n\rangle\langle\psi_n|$$

where the weights are: $q = \left(\sqrt{2}-1\right)/2 \quad p = \left(1-2q\right)/4$

We consider the operator: $A = H \otimes I + I \otimes H$

with $H = \mathrm{diag}(1,1,\text{-}1,\text{-}1)$

This state gives FQ = 9.3726 (where FQ_sep = 8).

Useful PPT states in metrology

How did we find the examples?

1) Naive approach. Let us recall the definition for the quantum Fisher information:

$$F_Q(\rho, A) = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k - \lambda_l} \left|\langle k | A | l \rangle\right|^2$$

where $\rho = \sum_k \lambda_k |k\rangle\langle k|$.

Let us optimize it over ρ_{PPT} states. However, it is a hard task to maximize a convex function over a convex set.

2) Instead, we use a heuristic search, the so-called see-saw method.

Useful PPT states in metrology

To this end, we note that the maximum for PPT states can alternatively be written as a double optimization:

$$\max_{\rho\in\mathrm{PPT}} F_Q(\rho, A) = \max_{\rho\in\mathrm{PPT}} \max_{M} \frac{\langle i[M, A]\rangle_\rho^2}{(\Delta M)^2}$$

The right-hand side expression comes from the relation with the Cramér-Rao bound, and the use of the error propagation formula.

Useful PPT states in metrology

See-saw procedure to maximize the precision: $\max_{\rho \in \mathrm{PPT}} \max_{M} \frac{\langle i[M, A] \rangle_{\rho}^{2}}{(\Delta M)^{2}}$

step 0: pick random operators *M*

Useful PPT states in metrology

See-saw procedure to maximize the precision: $\max\limits_{\rho\in\mathrm{PPT}} \max\limits_{M} \frac{\langle i[M,A]\rangle_\rho^2}{(\Delta M)^2}$

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step 1: maximize over PPT states ρ for a given *M*

⇒ this casts as an SDP

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⇨ *M* is given by the SLD, e.g. M.G. Paris 2009

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step 2: maximize over *M* for a given PPT state ρ

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step 3: back to step 1 until convergence is achieved

The FQ value cannot get worse with the number of iterations.

Summary

We have shown that PPT bound entangled are useful in overcoming the classical limit in quantum metrology and they can also be used to create Bell nonlocal correlations.

These results are based on the papers:

T. Vértesi, & N. Brunner (2014). Disproving the Peres conjecture by showing Bell nonlocality from bound entanglement. *Nature communications*, *5*, 5297.

G. Tóth, & T. Vértesi (2018). Quantum states with a positive partial transpose are useful for metrology. *Physical Review Letters*, *120*, 020506.

Thank you!