PAOLO PERINOTTI

INFORMATION, DISTURBANCE AND COMPATIBILITY

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NQSTI QST

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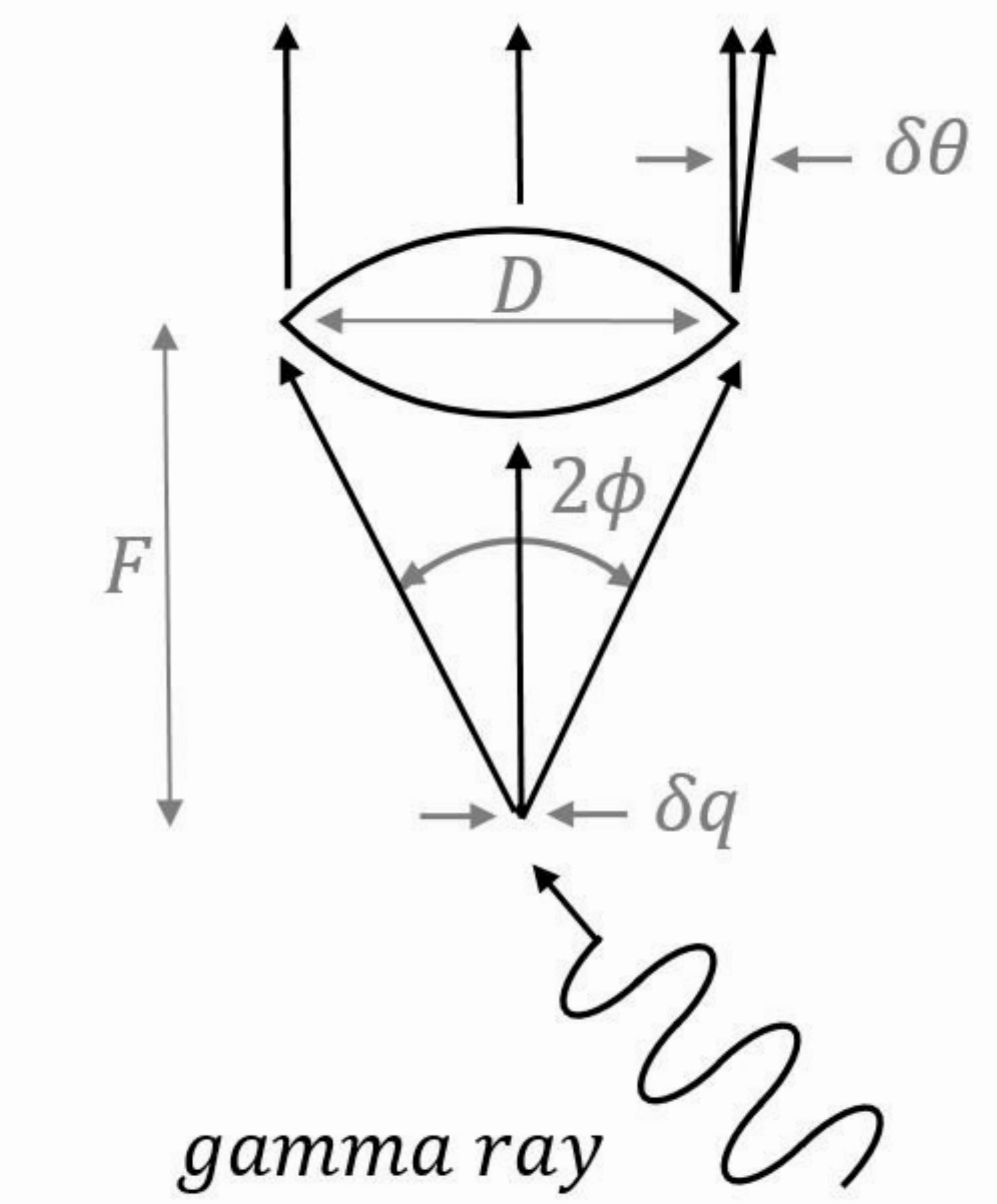
OUTLINE

- From Heisenberg to quantum information theory
- Widening the playground: Operational Probabilistic Theories
- Disturbance and correlations
- Information extraction
- (No) information without disturbance
- Compatibility: strong and weak
- Full compatibility
- MCT: no information without disturbance + full compatibility of observation tests

HEISENBERG'S GAMMA-RAY EXPERIMENT

Thought experiment used to justify intuitively the uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$



Statistical meaning: there is no quantum state such that most accurate predictions of x and p have

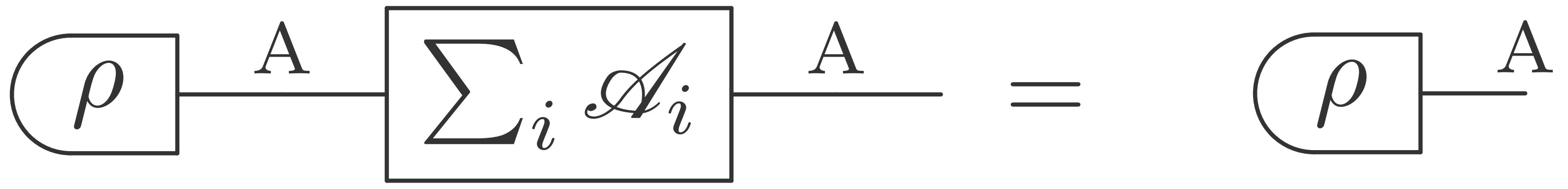
$$\Delta x \Delta p < \frac{\hbar}{2}$$

The thought experiment actually introduces two different but related problems:

1) position and momentum measurements are incompatible;
2) can we measure a system without disturbing its state?

NO-INFORMATION WITHOUT DISTURBANCE: QUANTUM

- In quantum information theory: definition by negation
  - Non-disturbing measurement: state after the measurement equal to the one before

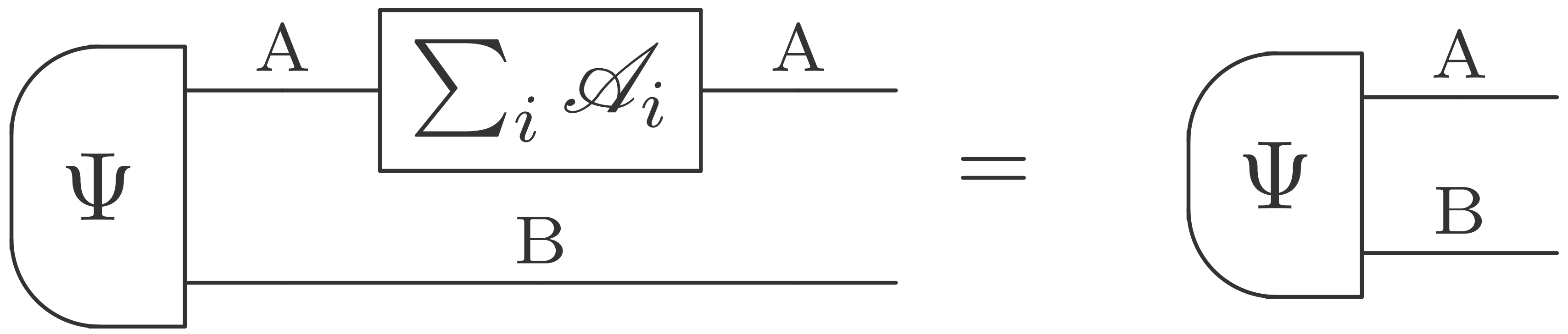


  - This is possible only if $\mathscr{A}_i = p_i \mathscr{I}$

- "No information without disturbance"

EQUIVALENT DEFINITION OF DISTURBANCE

- Equivalent notion of (no-)disturbance:



- Quantum information is quantum entanglement:

  "…we conclude that the deepest answer to the question is that quantum information lies in the entanglement between systems. Quantum communication, in this view, is fundamentally about the transfer of that entanglement from one system to another…"
  B. Schumacher and M. Westmoreland, "Quantum Processes, Systems & Information", Cambridge University Press (2010)

OPERATIONAL PROBABILISTIC THEORIES (IN A NUTSHELL)

System: 

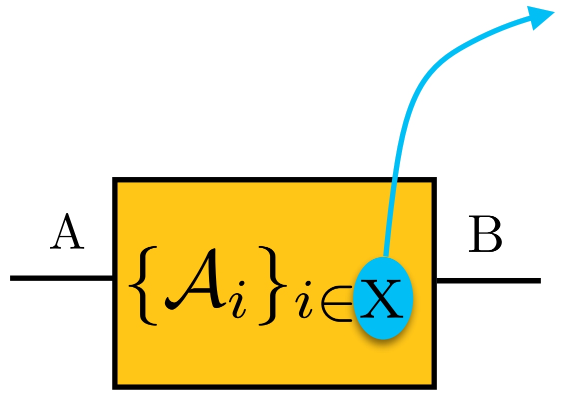
G. Chiribella, G. M. D'Ariano, and PP, Phys. Rev. A 81, 062348 (2010),
G. M. D'Ariano, G. Chiribella, and PP, Quantum Theory from first principles, CUP (2017)

OPERATIONAL PROBABILISTIC THEORIES (IN A NUTSHELL)

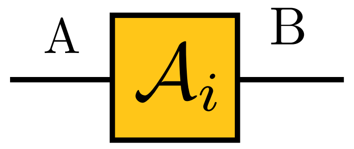
System:

finite set of
outcomes

Test:



collection of **events:**



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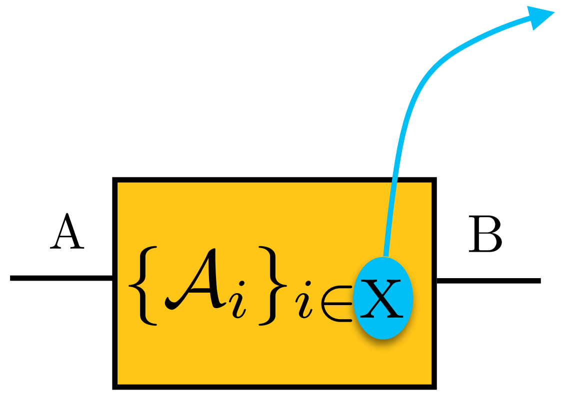

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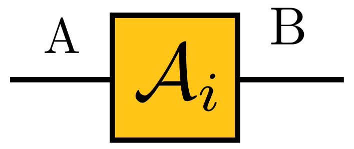
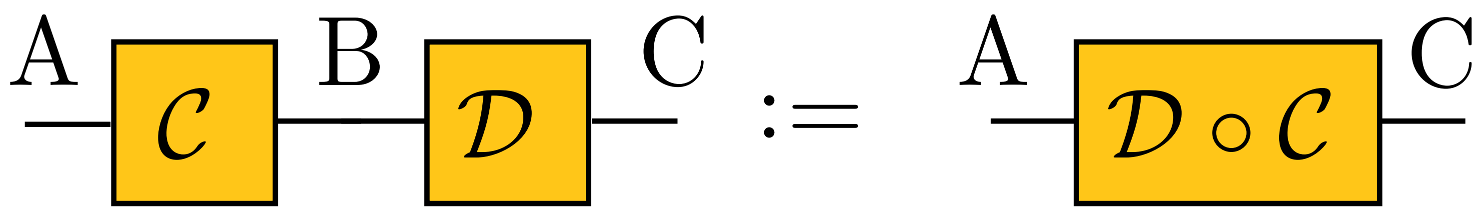


Composition:

in sequence



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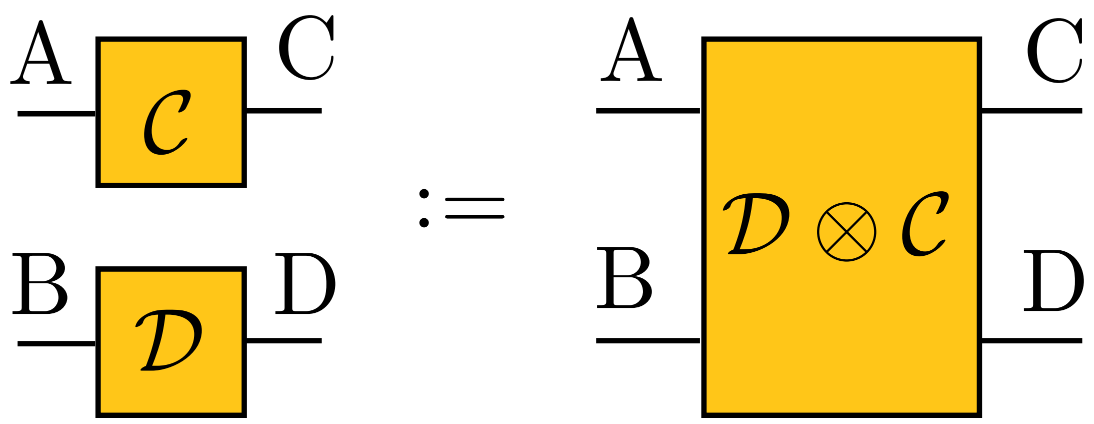

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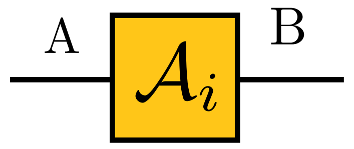
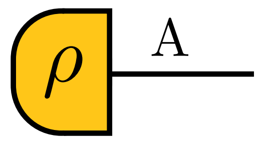
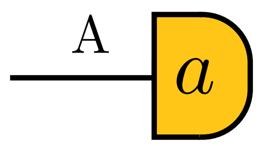


Preparation test: collection of **states**



Observation test: collection of **effects**



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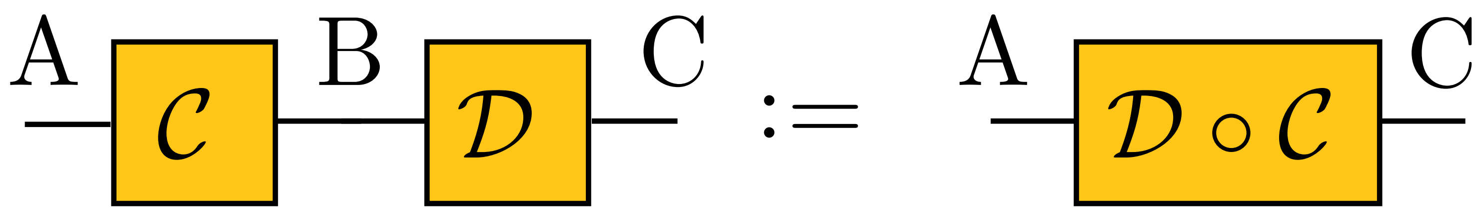

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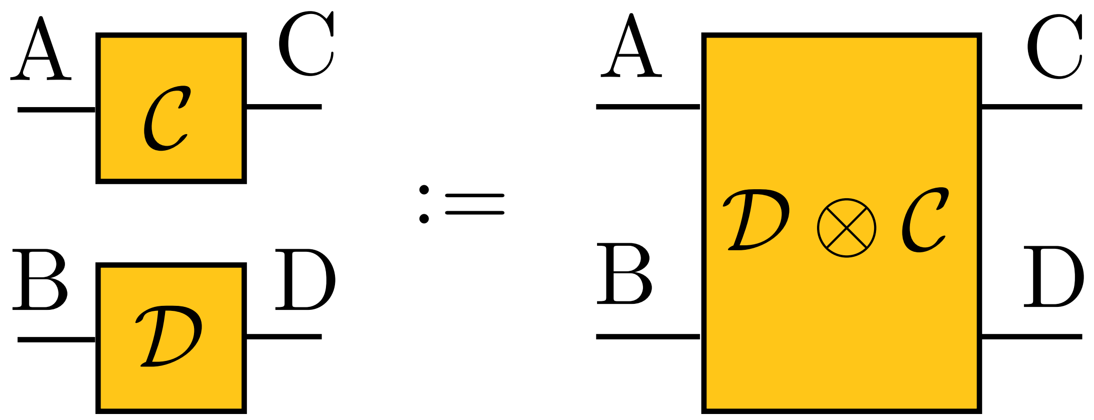

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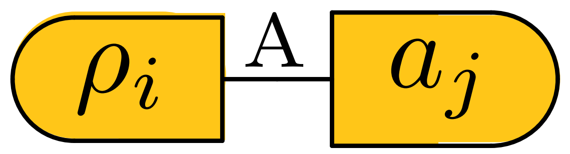
PROBABILISTIC STRUCTURE

Probabilistic structure:



$:= \mathrm{Pr}[i, j]$

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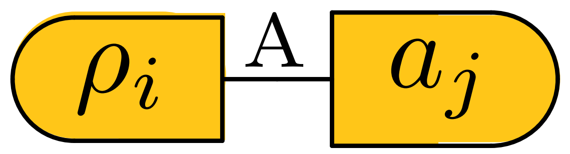

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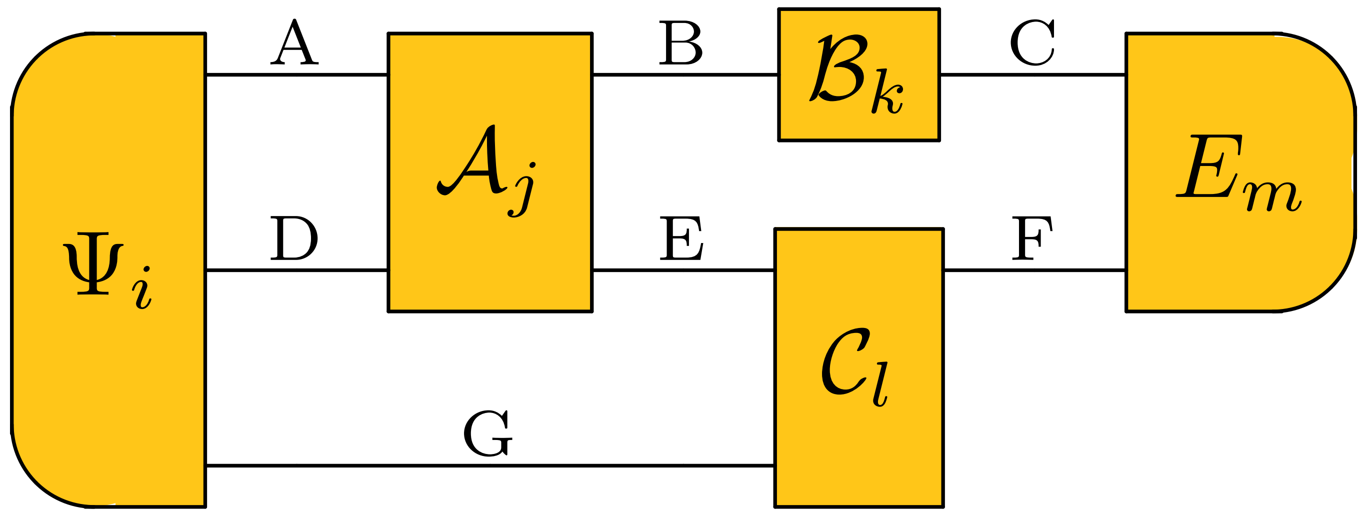

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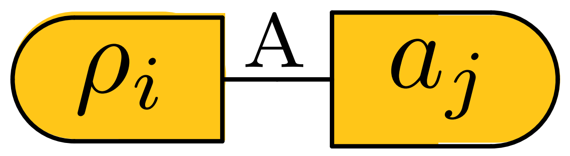

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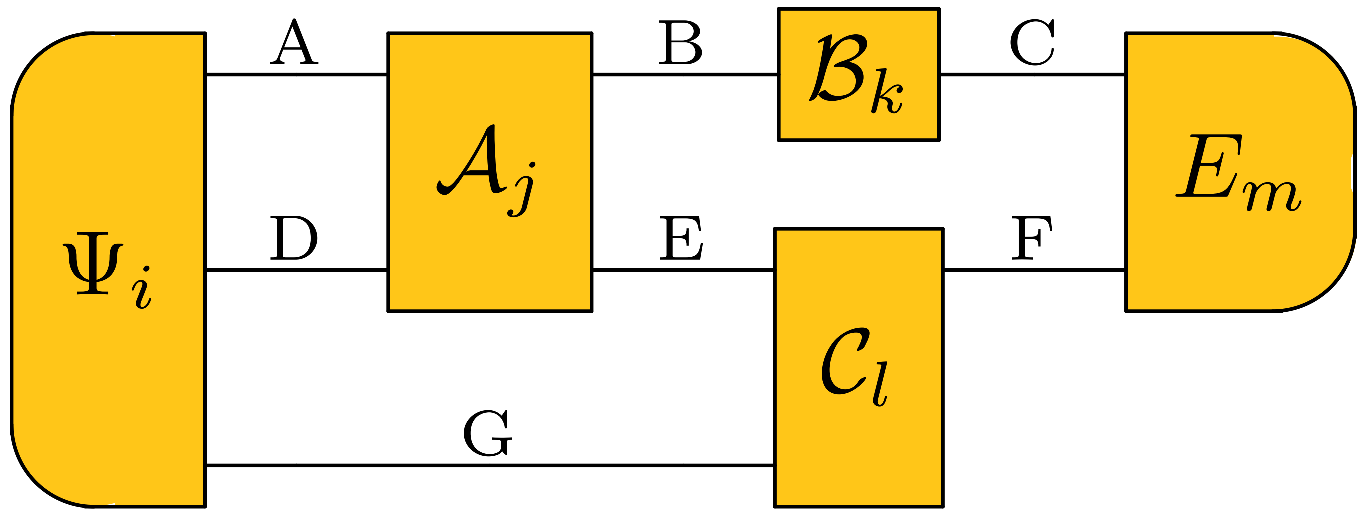

$= \Pr[i, j, k, l, m]$

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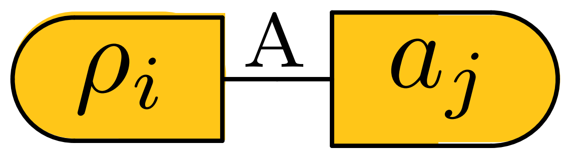

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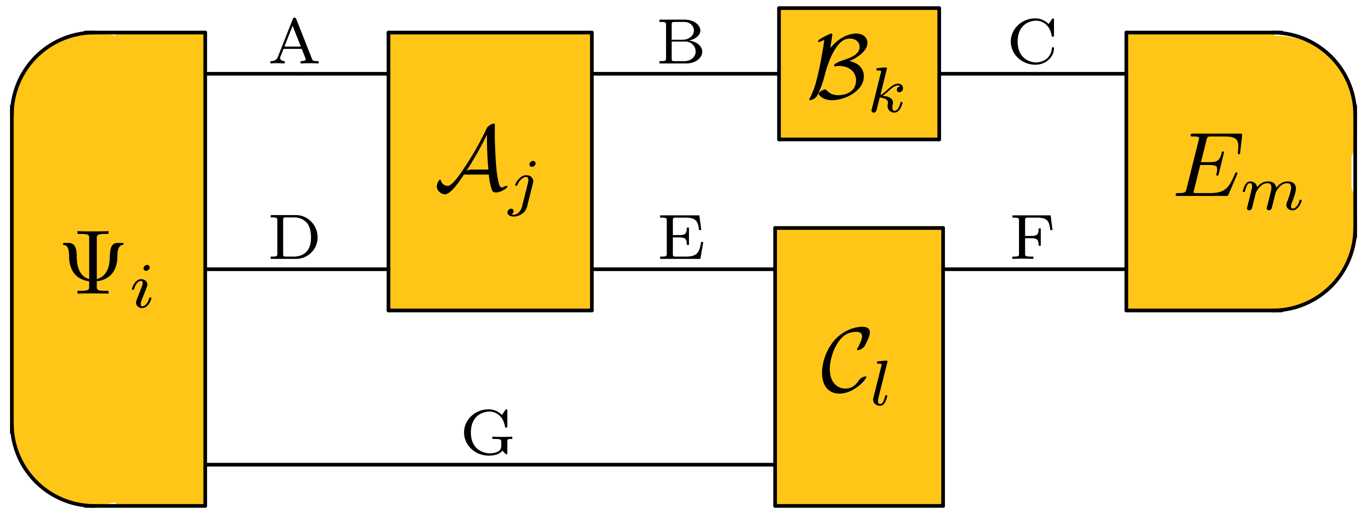

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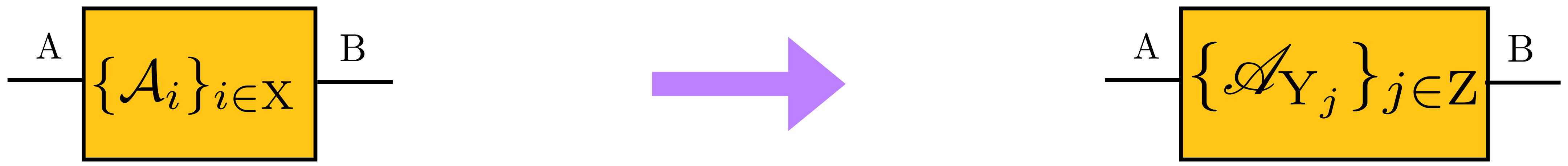
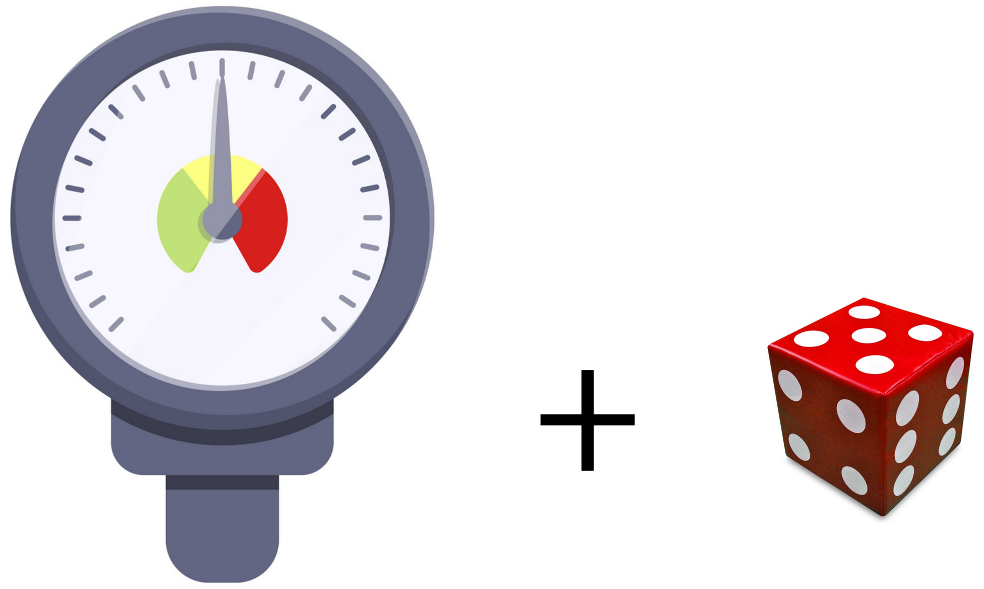


Coarse graining: $\forall j \in Z\ Y_j \subseteq X, \quad j_1 \neq j_2 \Rightarrow Y_{j_1} \cap Y_{j_2} = \emptyset, \quad \bigcup_j Y_j = X \quad \Rightarrow \quad \exists\{\mathscr{A}_{Y_j}\}_{j\in Z}$

ATOMIC EVENTS

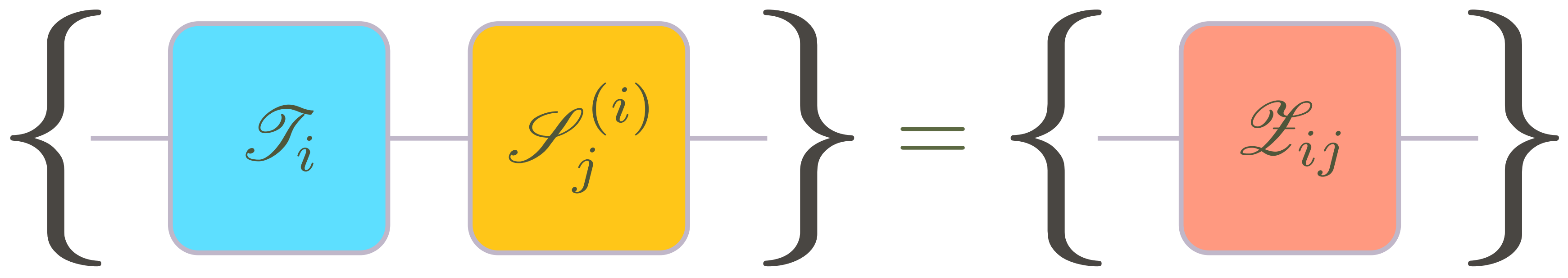
- Atomic event: an event that can be refined only trivially

$$\mathscr{A} = \sum_j \mathscr{A}_j \quad \Rightarrow \quad \mathscr{A}_j = p_j \mathscr{A}$$



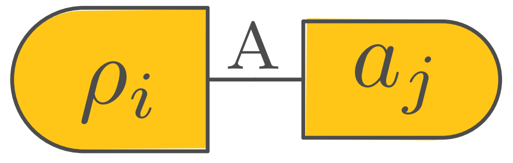
CAUSAL THEORIES

- **Strong causality**: arbitrary conditioning



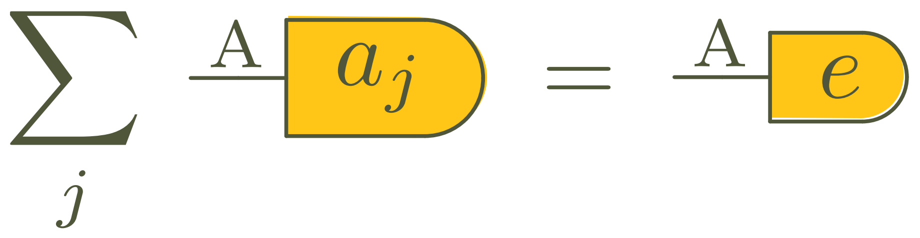
  - Implies **weak causality**:

$$p_a(\rho_i) := \sum_j$$



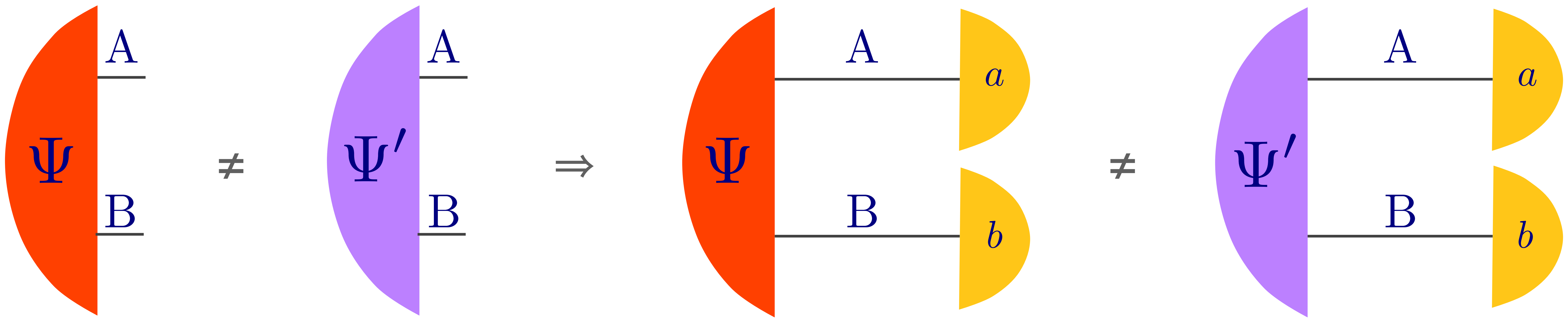
$$= p(\rho_i)$$

    - Uniqueness of the deterministic effect



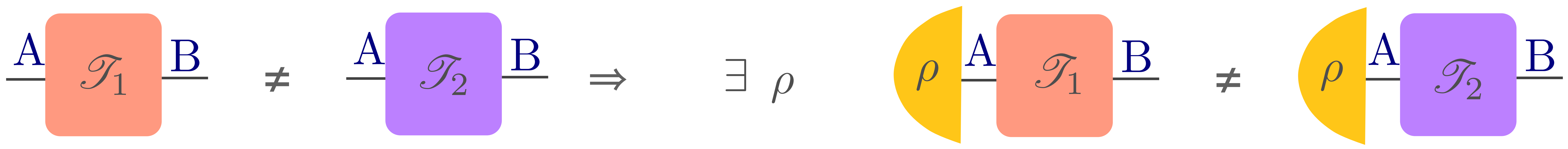
LOCAL DISCRIMINABILITY

- **Local discriminability**: it is possible to distinguish bipartite states by local observations



LOCAL DISCRIMINABILITY

- Local discriminability ⇒ transformations ↔ local action



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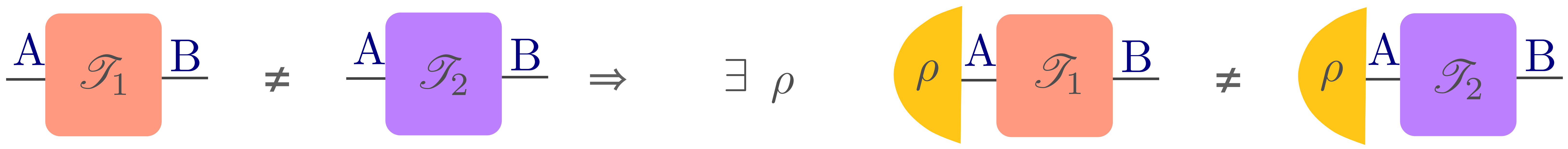
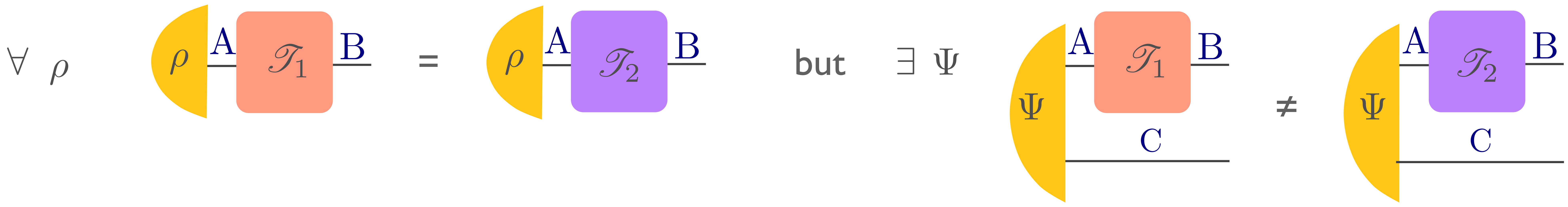
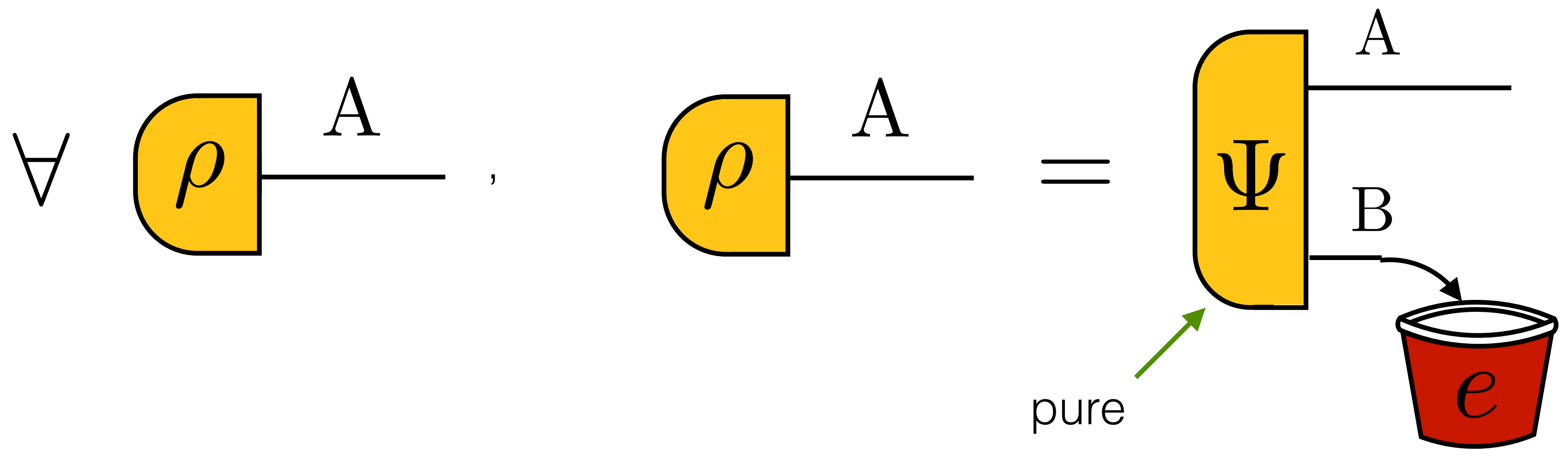


- No local discriminability ⇒ it can happen that

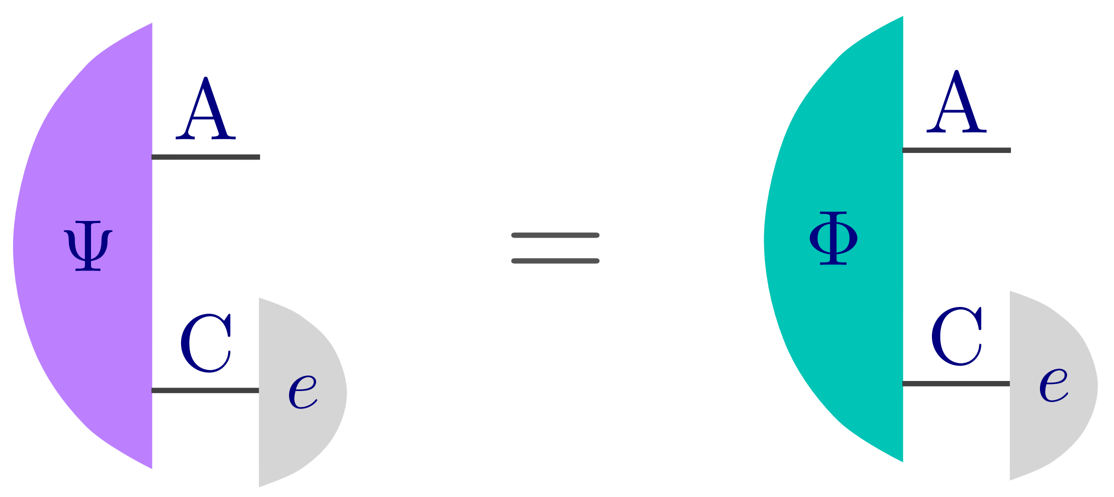


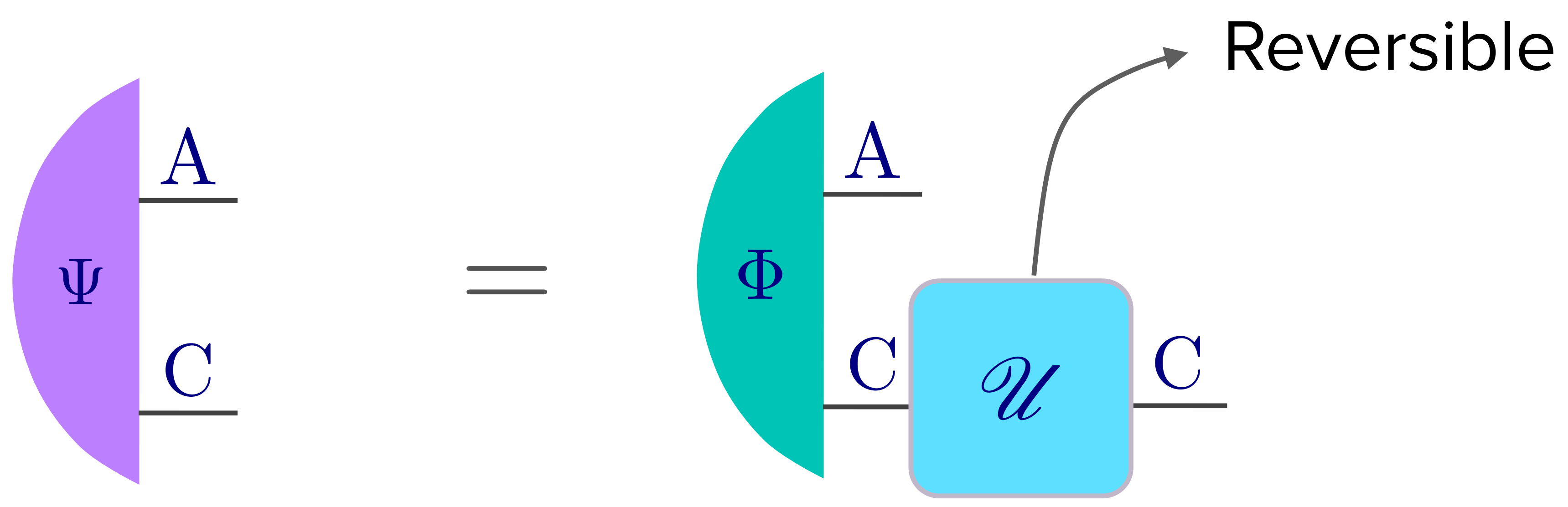
PURIFICATION

- Existence of purification:



- Uniqueness of purification



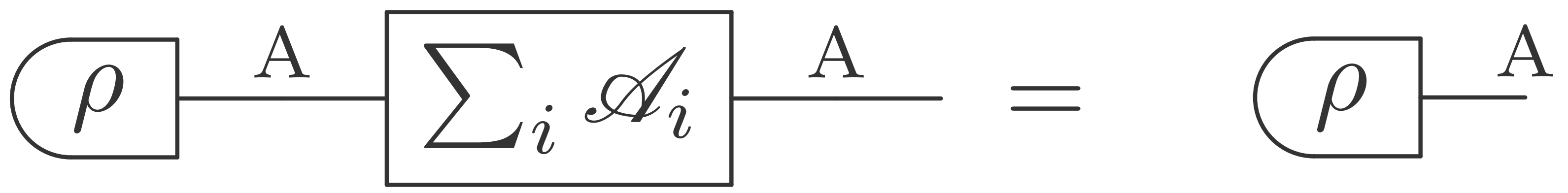


INFORMATION AND DISTURBANCE

DISTURBANCE OF CORRELATIONS

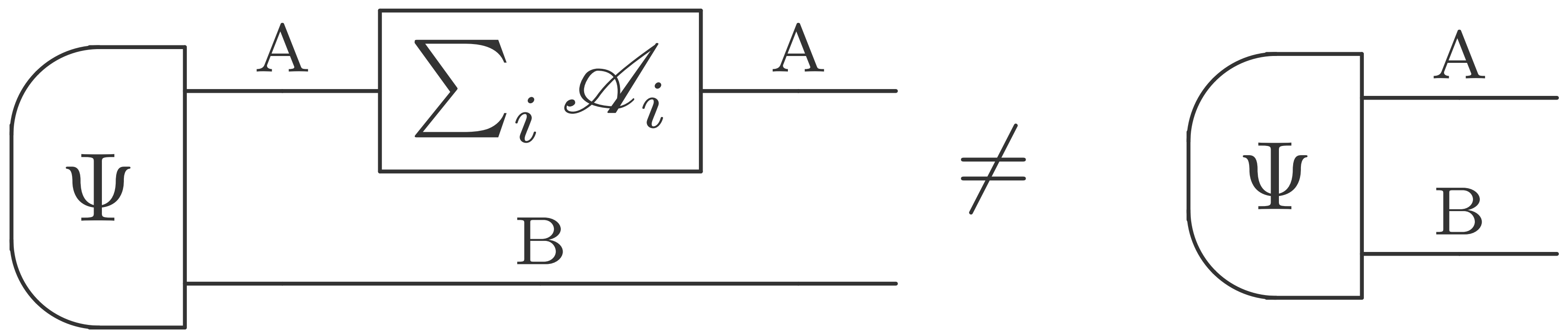
- (No-)Disturbance on correlations: the definition for general theories
- There are indeed situations where

$\forall \rho$



but

$\exists \Psi$



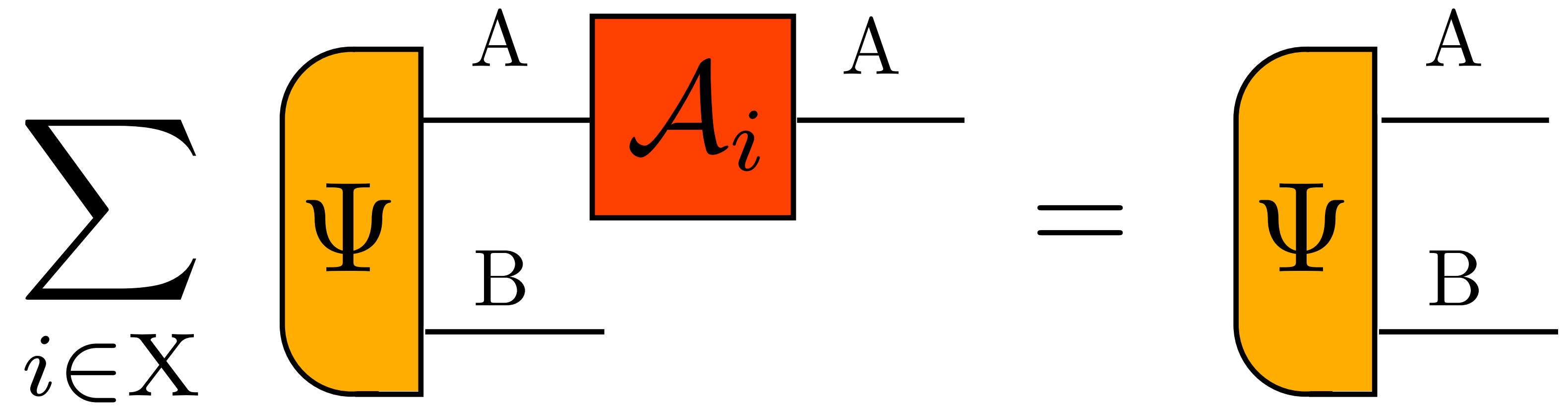
NON-DISTURBING TESTS

- The usual definition is inadequate in the absence of local discriminability

NON-DISTURBING TESTS

- The usual definition is inadequate in the absence of local discriminability
- Definition (non-disturbing test): $\{\mathcal{A}_i\}_{i\in \mathrm{X}}$ is non-disturbing if

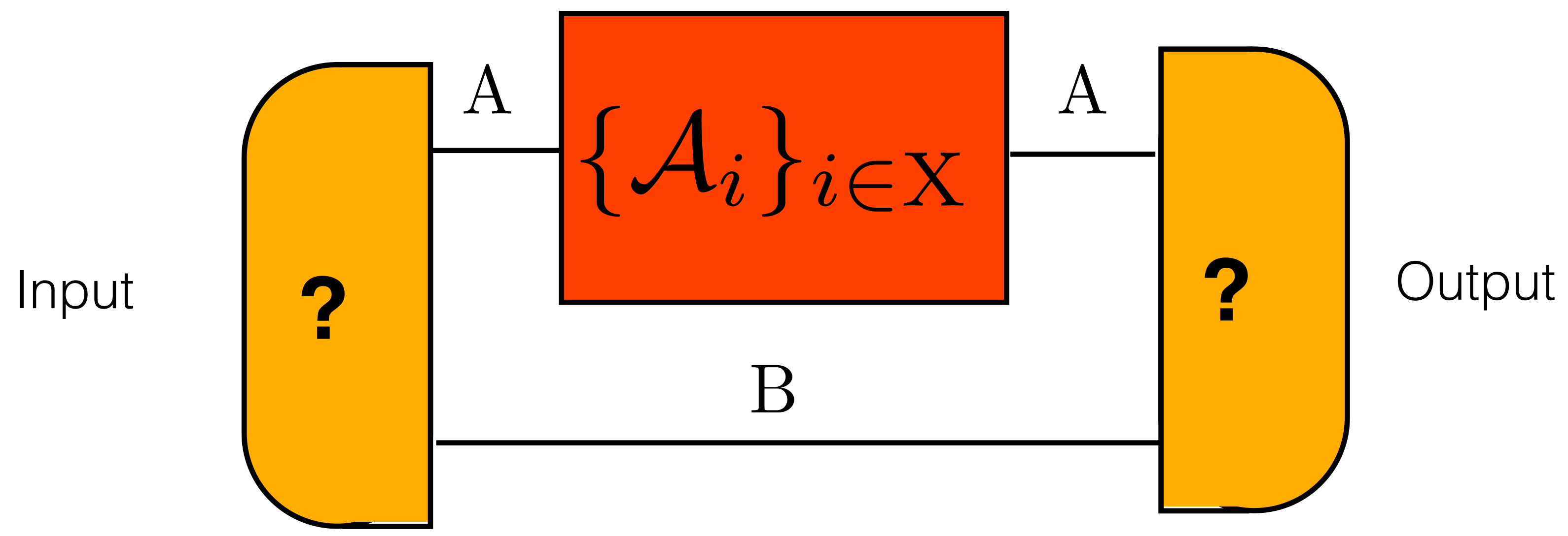
$\forall \mathrm{B}$



$\forall\, \Psi \in \mathrm{St}(\mathrm{AB})$

INFORMATION FROM A TEST

- Consider a test of a theory



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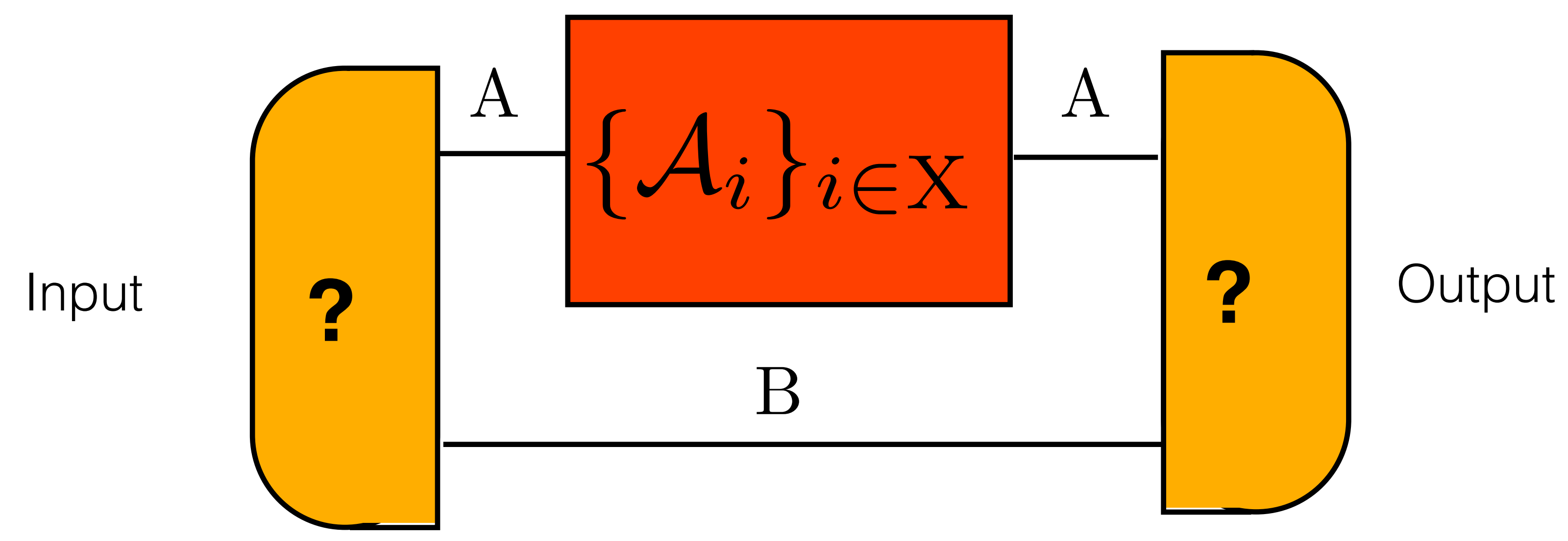

- When does the test provide info on the input?

INFORMATION FROM A TEST

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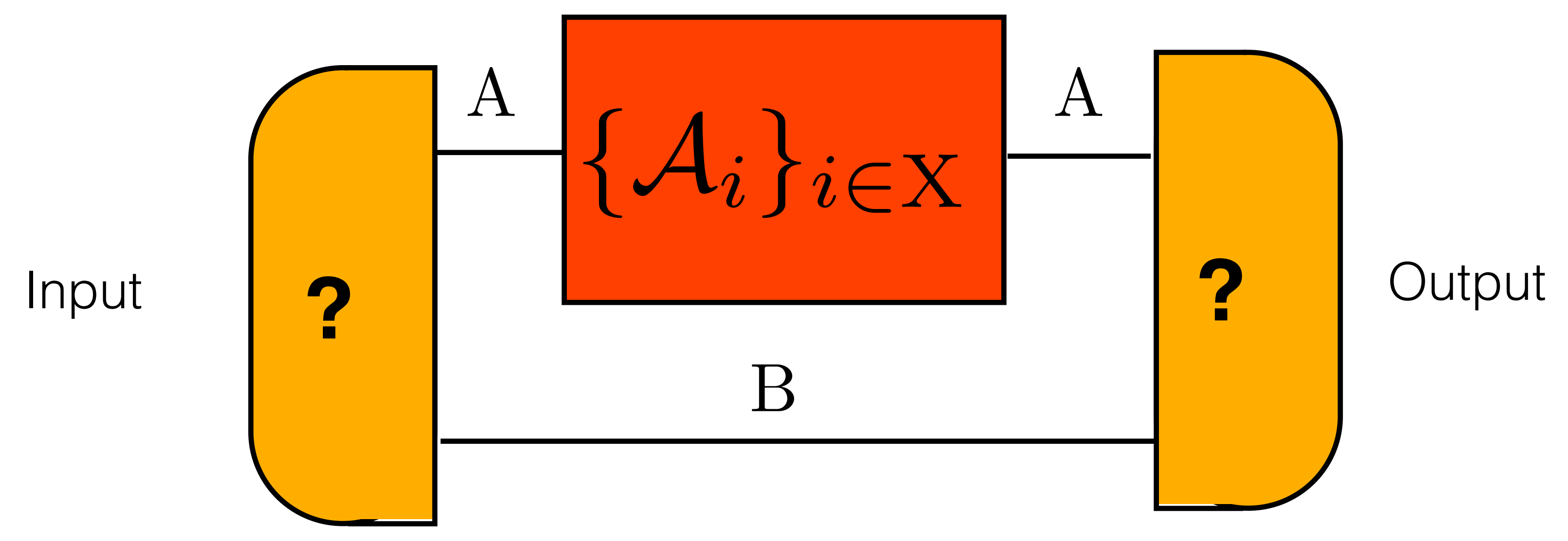

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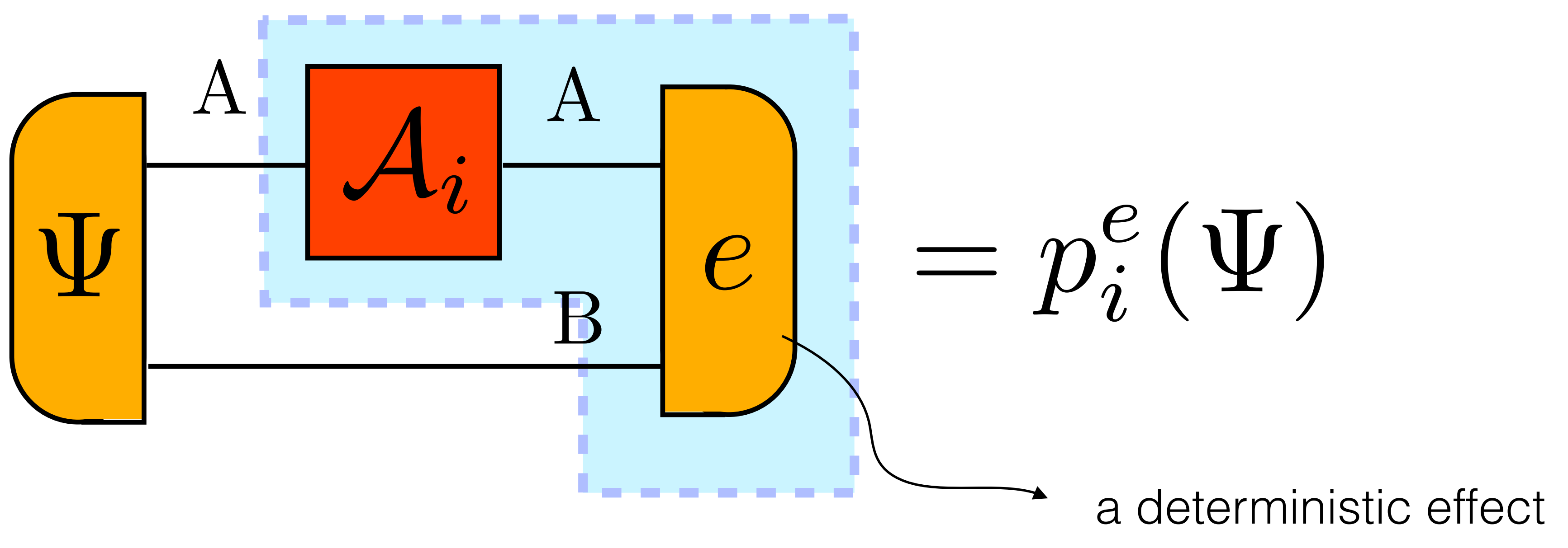

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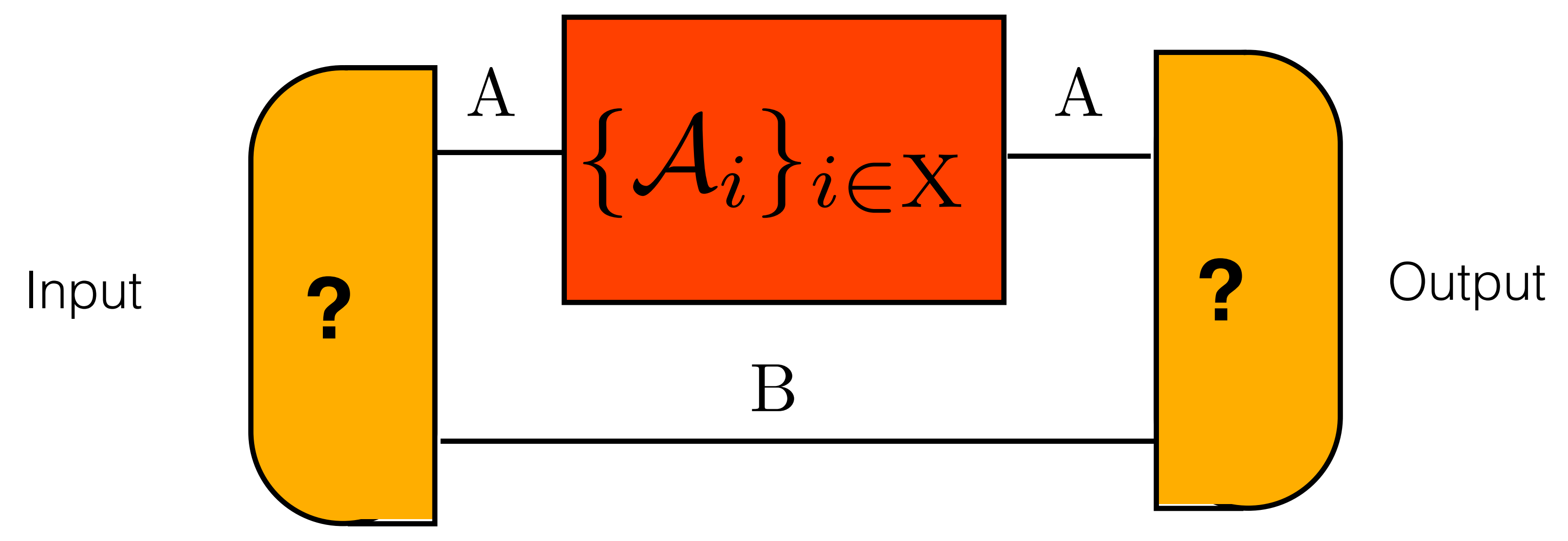

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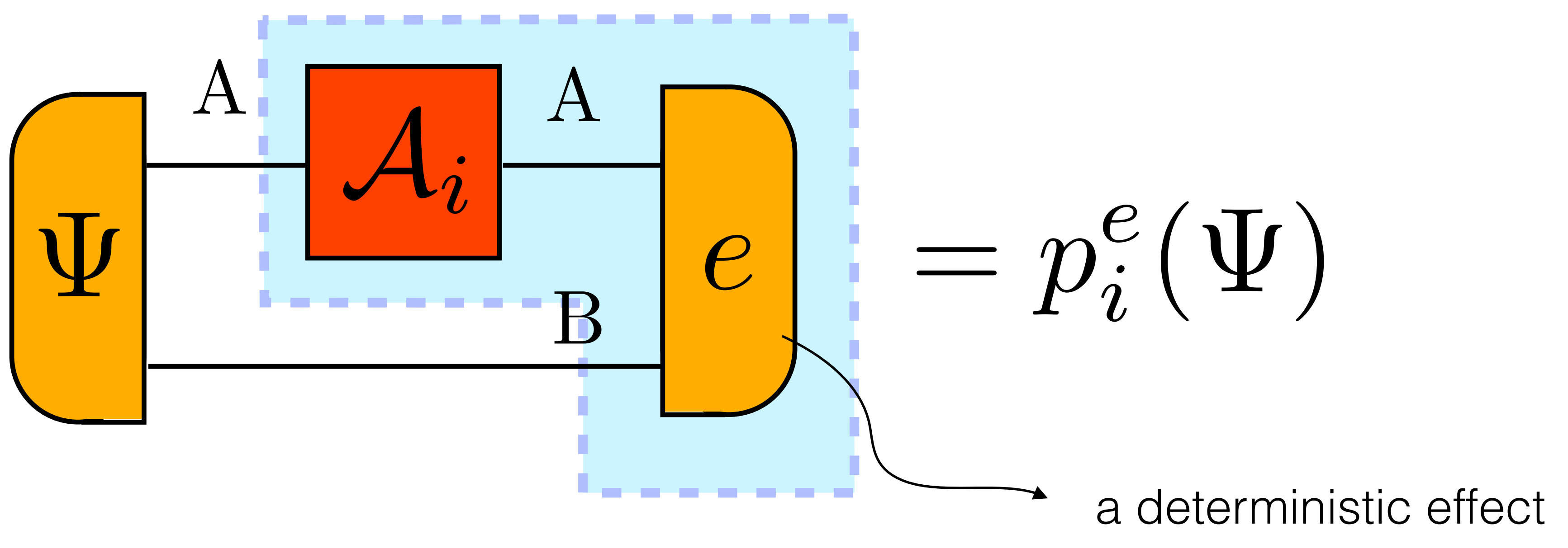
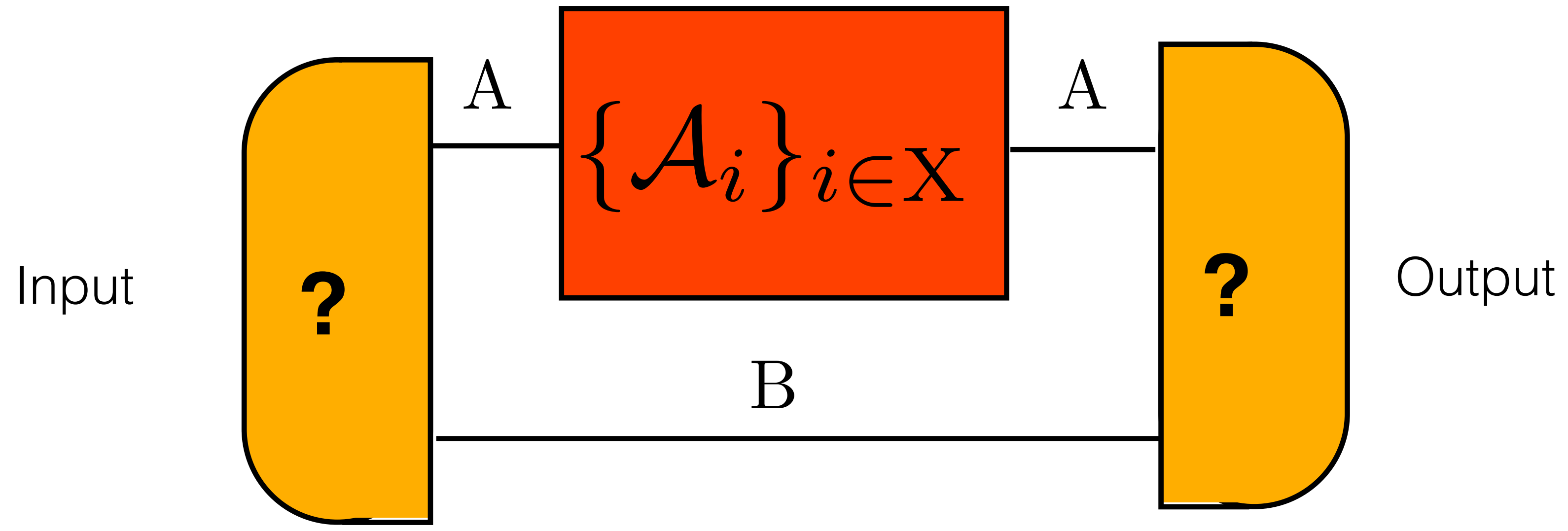


- When does the test provide info on the output?

INFORMATION FROM A TEST

- Consider a test of a theory



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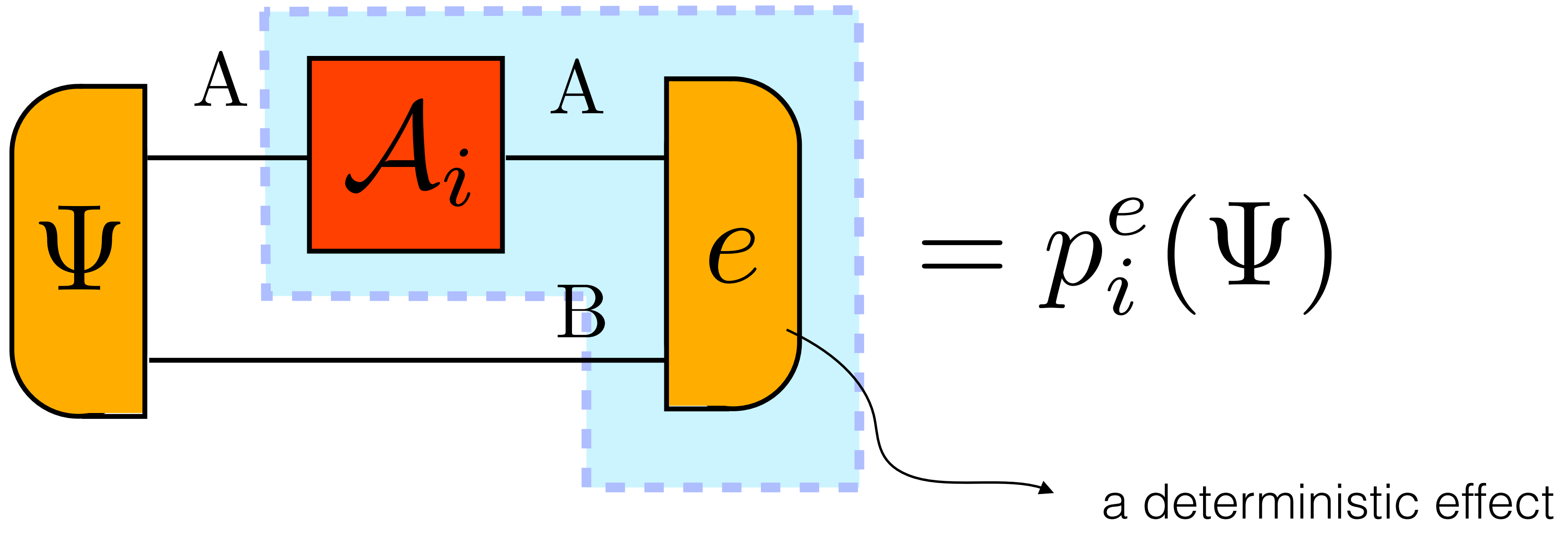
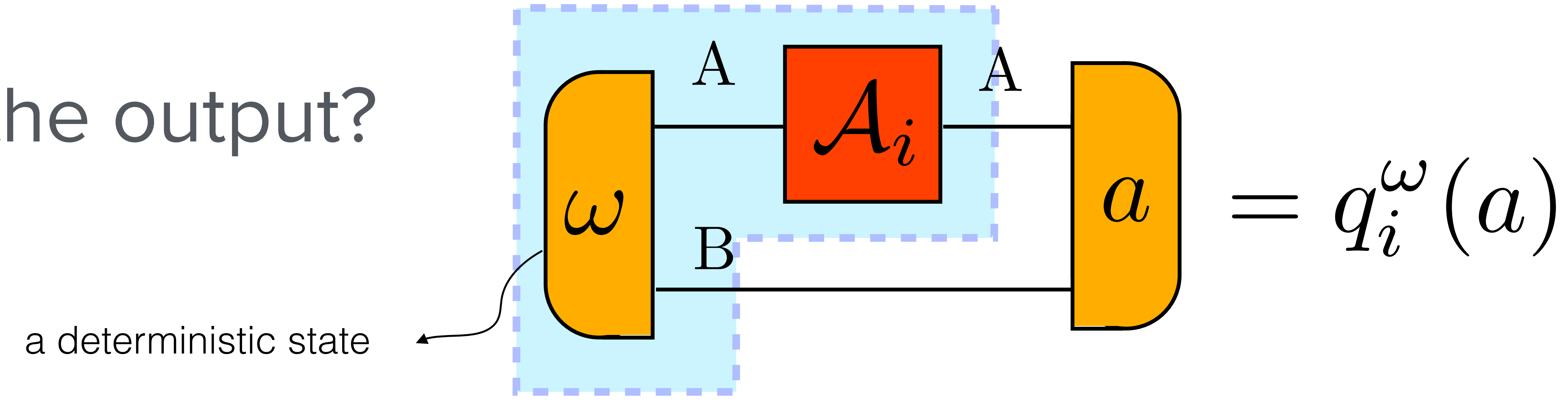


- When does the test provide info on the output?



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we say that **it does not provide information** if

G. M. D'Ariano, PP, and A. Tosini, Quantum 4, 363 (2020).

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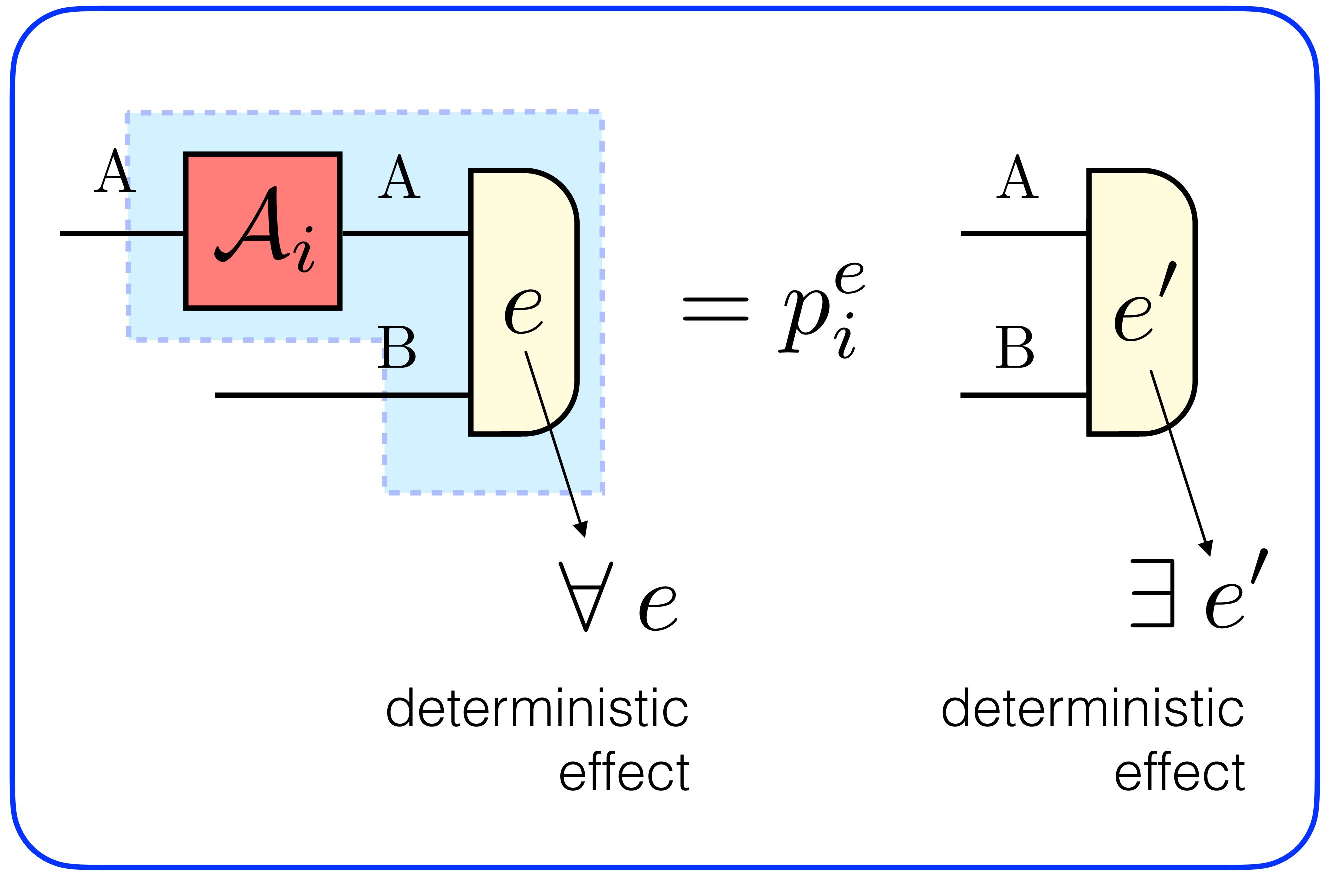

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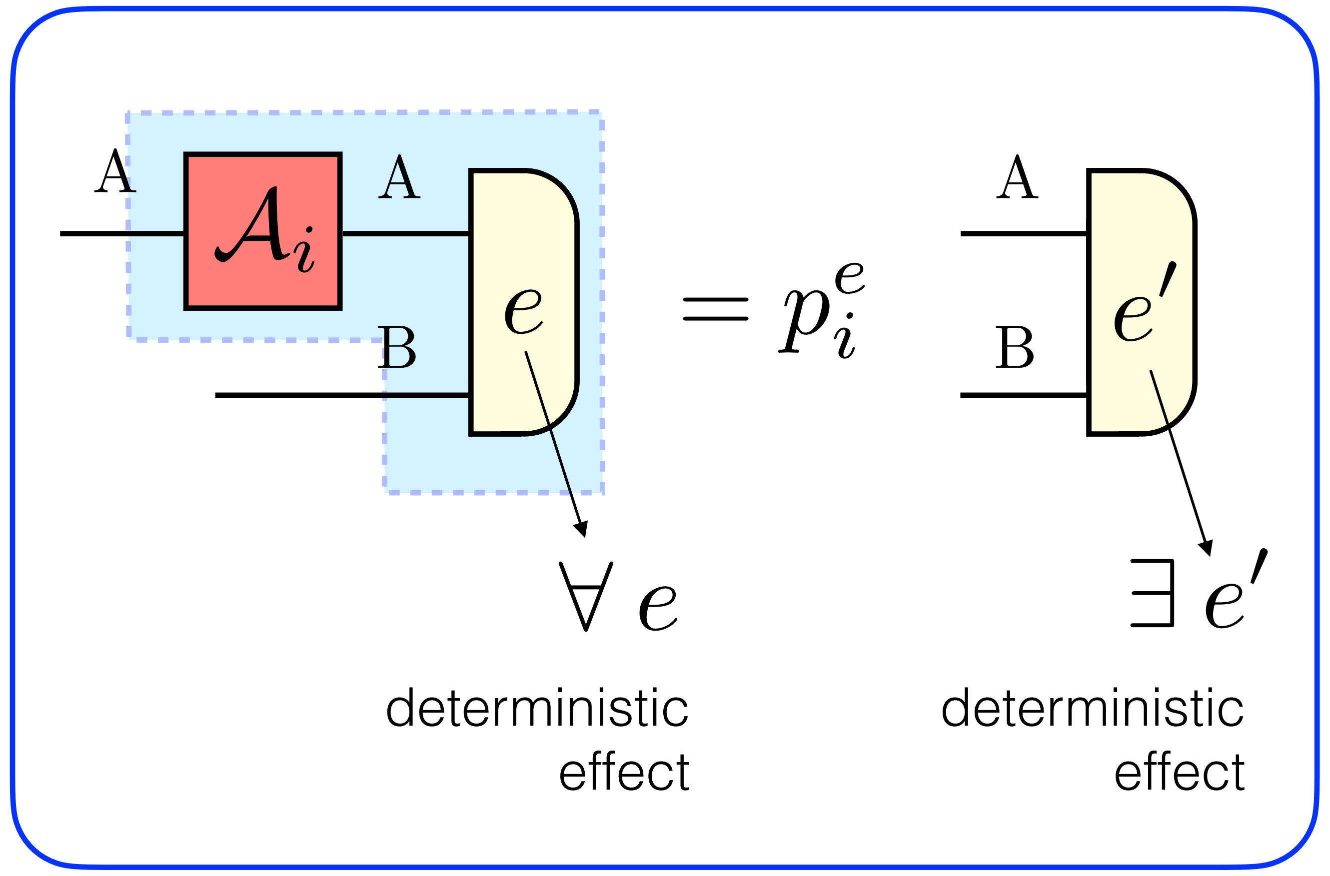
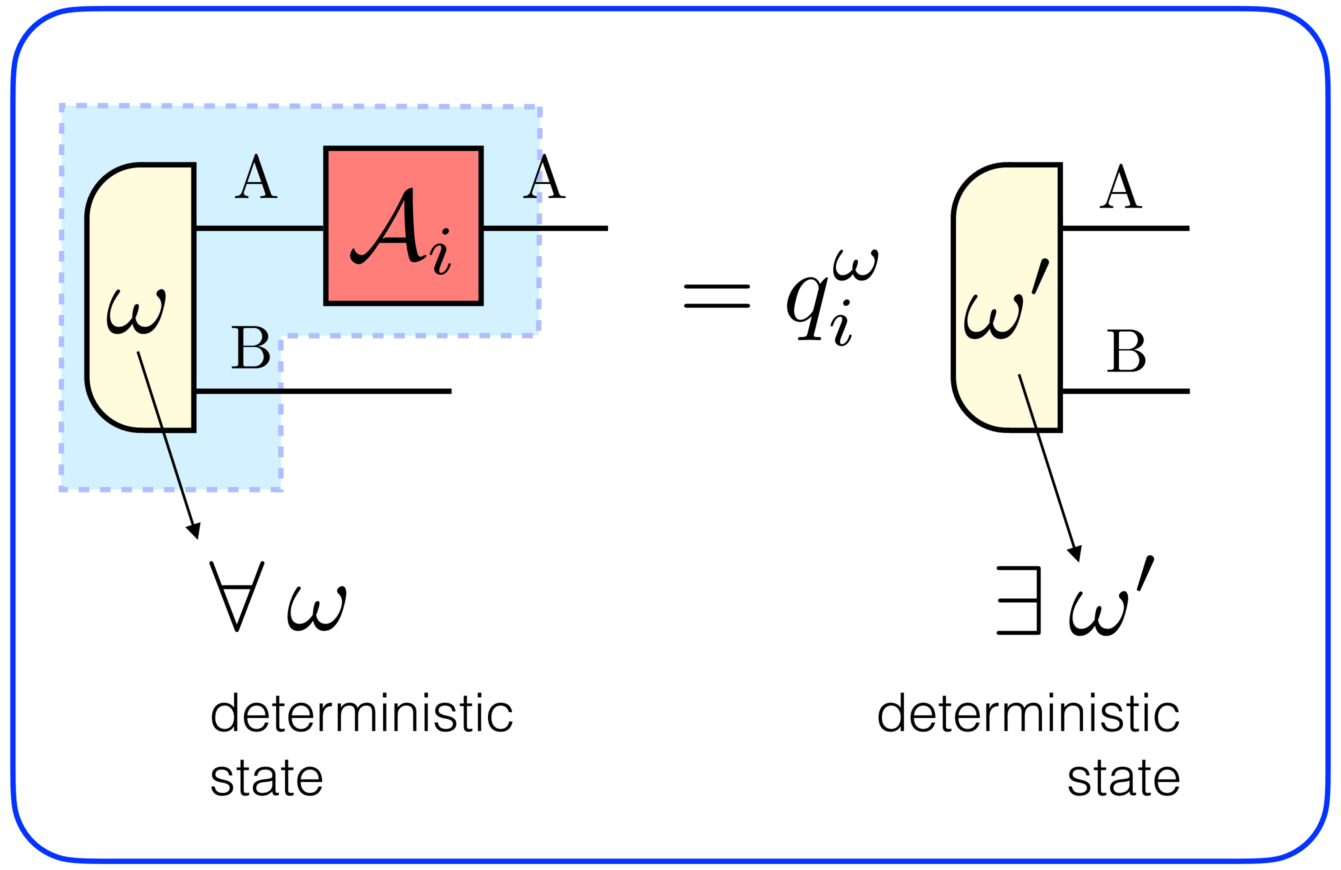


and

no-information on the output



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NO INFORMATION WITHOUT DISTURBANCE

- We say that a theory has no information without disturbance if

$$\{\mathcal{A}_i\}_{i\in\mathrm{X}} \quad \textbf{non-disturbing} \Rightarrow \quad \{\mathcal{A}_i\}_{i\in\mathrm{X}} \quad \textbf{no-information}$$

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- Theorem: a theory has NIWD iff the identity transformation is atomic for every system

$$\mathcal{I}_{\mathrm{A}} = \sum_i \mathcal{A}_i \Rightarrow \mathcal{A}_i \propto \mathcal{I}_{\mathrm{A}}$$

G. M. D'Ariano, PP, and A. Tosini, Quantum 4, 363 (2020).

OTHER CONDITIONS FOR N.I.W.D.

- A theory has NIWD $\Leftrightarrow$ for every system there exists a reversible atomic transformation
- Sufficient condition for NIWD: convexity + existence of purification

G. M. D'Ariano, PP, and A. Tosini, Quantum 4, 363 (2020).

INFORMATION WITHOUT DISTURBANCE

- What if the identity map is not atomic?

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- Information without disturbance: classical information
- Decomposition of the identity ⇒ decomposition of the sets of states and effects

$$\mathsf{St}(\mathrm{A}) = \bigoplus_i \mathsf{St}_i(\mathrm{A})$$

$$\mathsf{Eff}(\mathrm{A}) = \bigoplus_i \mathsf{Eff}_i(\mathrm{A})$$

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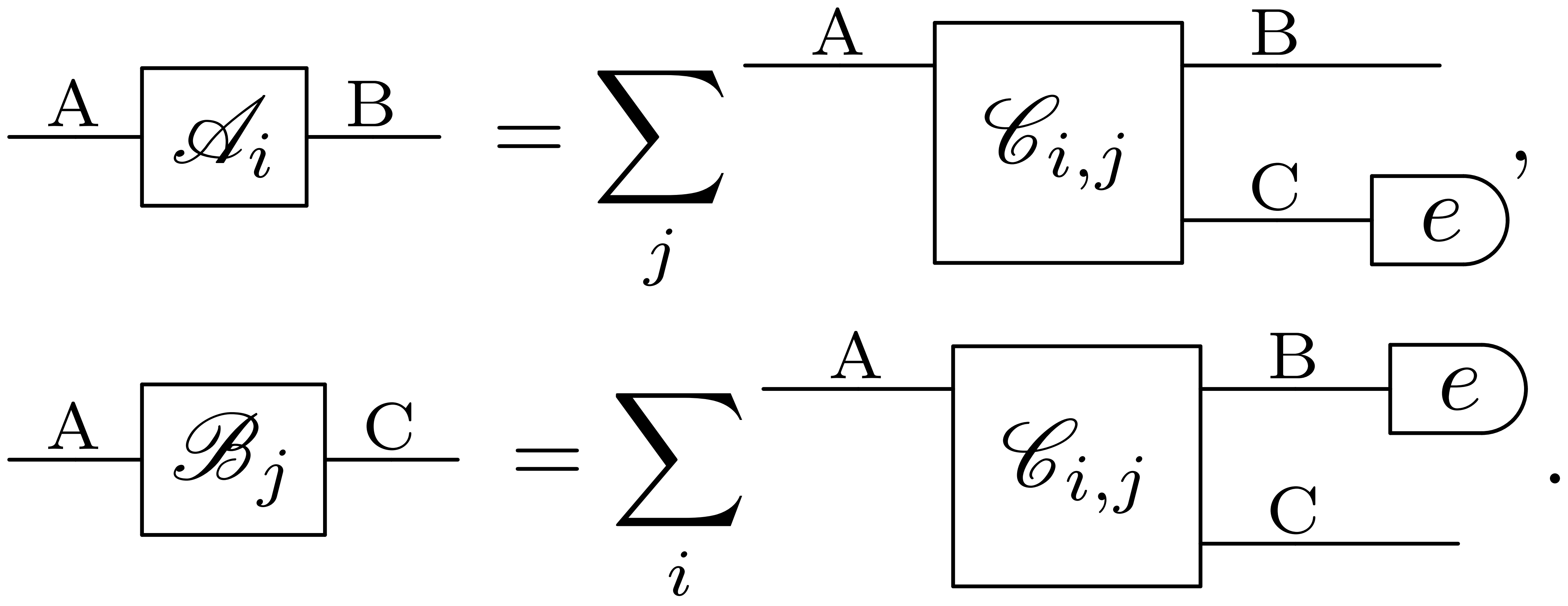
COMPATIBILITY OF TESTS

COMPATIBILITY OF OBSERVATION-TESTS

- Question: what does it mean for two tests to be compatible?
- For observation tests: widely studied in the quantum literature
  - POVMs $\{P_i\}_{i \in \mathrm{X}}$ and $\{Q_j\}_{j \in \mathrm{Y}}$ are compatible if there exists a POVM $\{R_{i,j}\}_{(i,j) \in \mathrm{X} \times \mathrm{Y}}$ s.t.
    - $P_i = \sum_{j \in \mathrm{Y}} R_{i,j}$
    - $Q_j = \sum_{i \in \mathrm{X}} R_{i,j}$
- Possibility to gather information about both outcomes in a single experiment

STRONG COMPATIBILITY OF TESTS

- Definition 1: **Strong compatibility** (mimicking compatible observation-tests)
  - $\{\mathscr{A}_i\}_{i\in\mathrm{X}} : \mathrm{A} \to \mathrm{B}$ and $\{\mathscr{B}_j\}_{j\in\mathrm{Y}} : \mathrm{A} \to \mathrm{C}$ are **strongly compatible** if there exists $\{\mathscr{C}_{i,j}\}_{(i,j)\in\mathrm{X}\times\mathrm{Y}} : \mathrm{A} \to \mathrm{BC}$ such that



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WEAK COMPATIBILITY OF TESTS

- Tests: not only statistics, but also evolution

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- Tests: not only statistics, but also **evolution**
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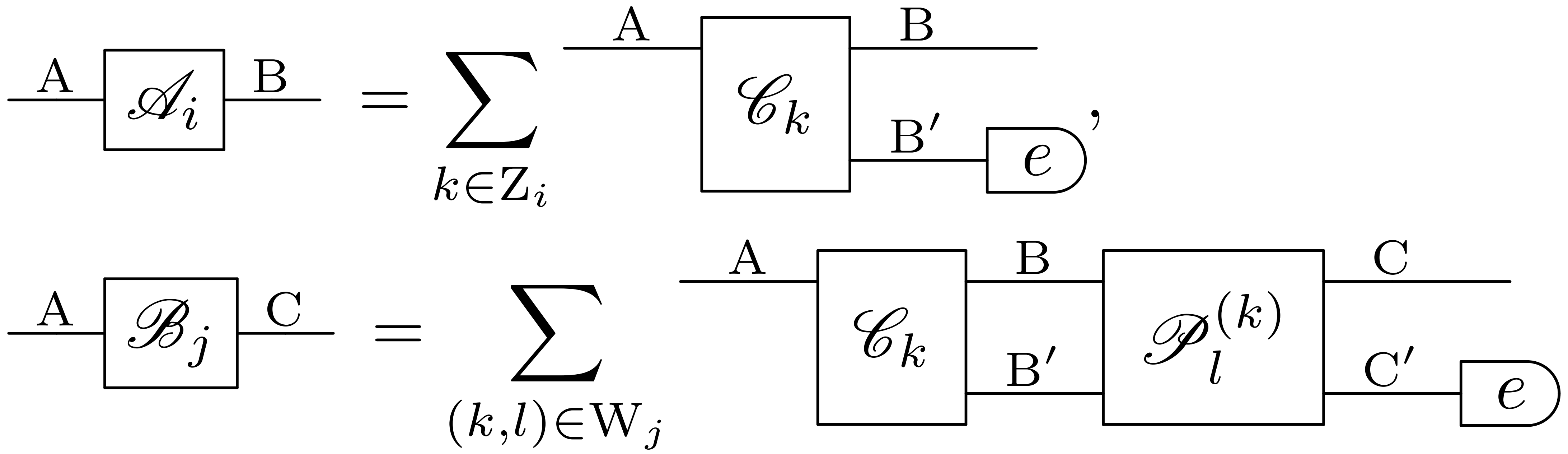
- $\{\mathscr{A}_i\}_{i\in \mathrm{X}} : \mathrm{A} \to \mathrm{B}$ **does not exclude** $\{\mathscr{B}_j\}_{j\in \mathrm{Y}} : \mathrm{A} \to \mathrm{C}$ if



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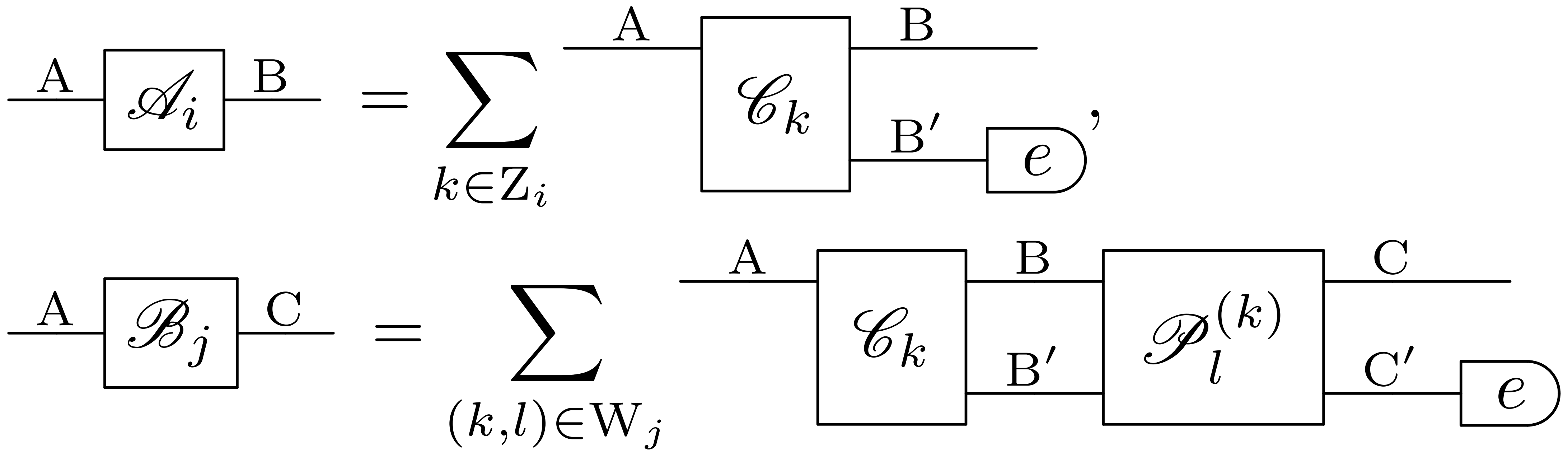

- Definition 2: **Weak compatibility**
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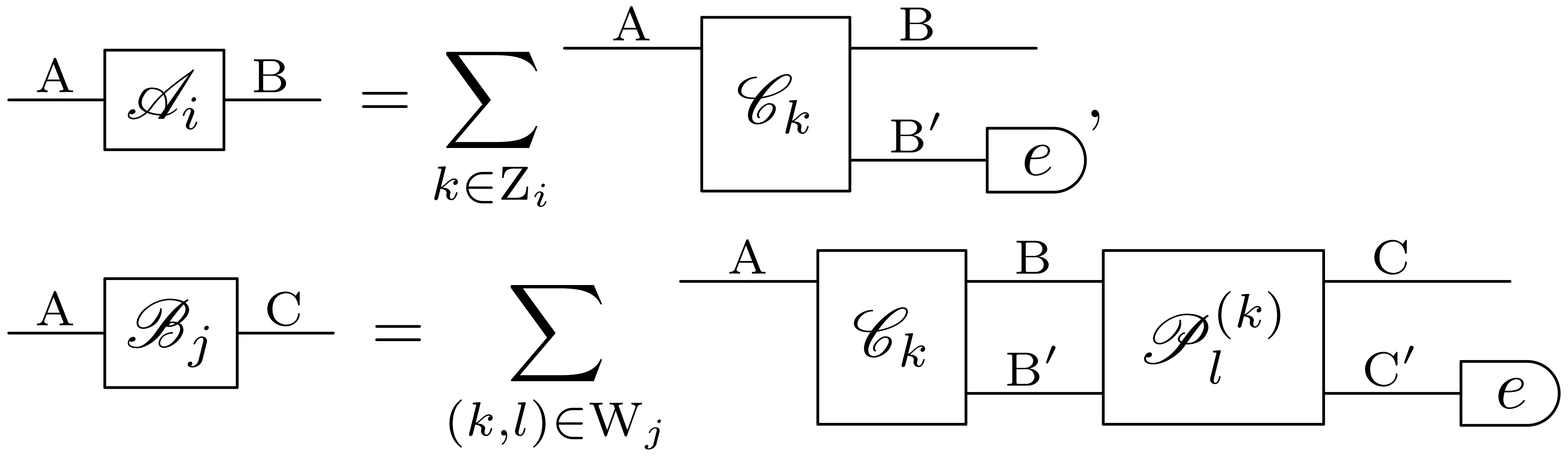

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QUANTUM CHANNELS

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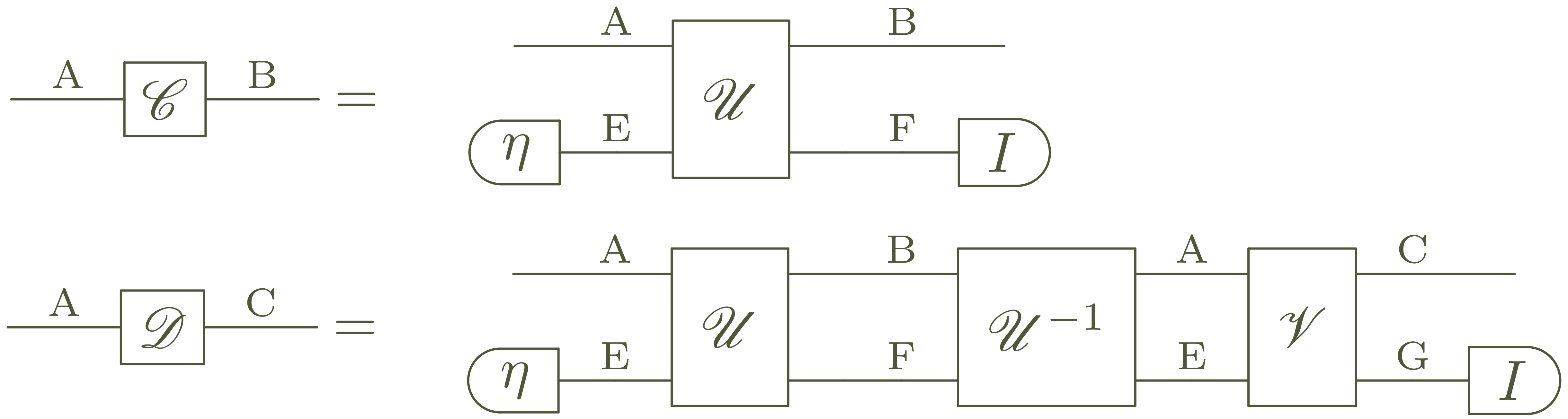

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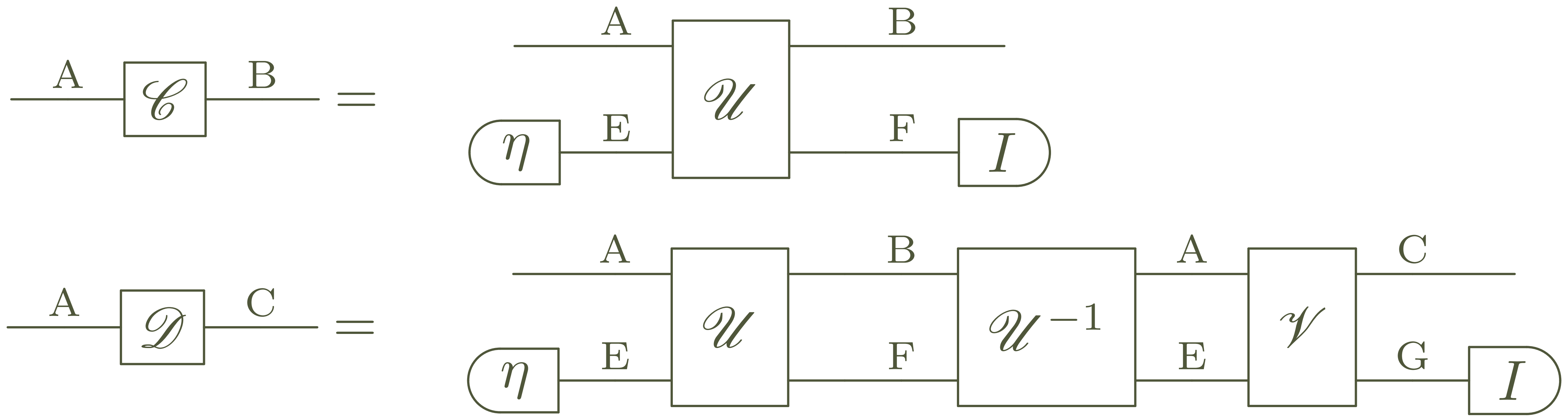

- **Quantum instruments are not weakly compatible**

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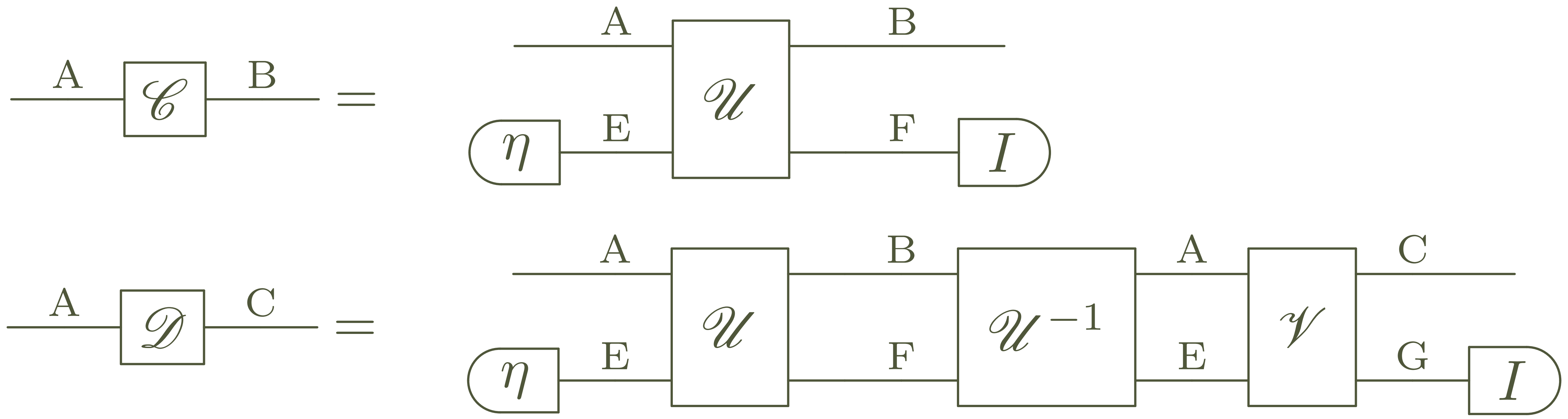

- Quantum instruments are not weakly compatible
- Irreversibility by discarding ancillary systems (entails weak compatibility)

  is radically different from

  irreversibility by gathering information (entails weak incompatibility)

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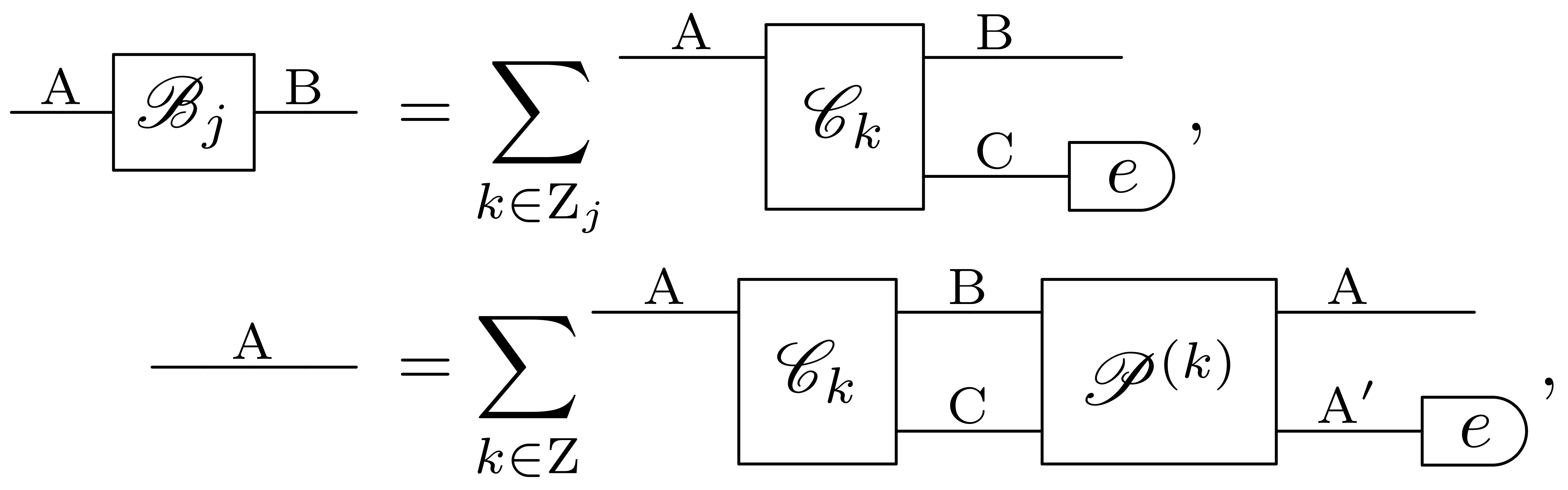
FULL COMPATIBILITY

- **Definition**: A theory has **full compatibility** if every two tests are weakly compatible
- **Lemma**: A theory has full compatibility iff every test does not exclude the identity
  - In this case we say that the theory has **full-information without disturbance**



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UNCERTAINTY VS INCOMPATIBILITY

FULL INFORMATION WITHOUT DISTURBANCE AND FULL COMPATIBILITY

- **Full compatibility holds if and only if full information without disturbance does**

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FULL INFORMATION WITHOUT DISTURBANCE AND FULL COMPATIBILITY

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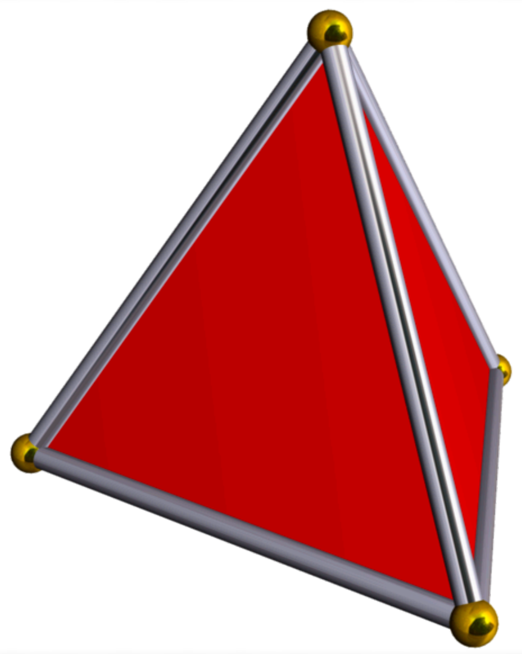
- Necessary condition: every system has a "classical state space"

$$\mathsf{St}(\mathrm{A}) = \bigoplus_i \mathsf{St}_i(\mathrm{A})$$

$$\dim \mathsf{St}_i(\mathrm{A}) = 1$$



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- **The composition rule and transformations are not necessarily classical**

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COMPATIBILITY

- **Is a classical state space sufficient for full compatibility?**

M. Erba, PP, D. Rolino and A. Tosini, arXiv:2305.16931

COMPATIBILITY

- Is a classical state space sufficient for full compatibility?
  - One might build a theory **with no information without disturbance**, though **without uncertainty**

F.I.W.D. ⟷ Full (weak) compatibility

↓

Classical state space

M. Erba, PP, D. Rolino and A. Tosini, arXiv:2305.16931

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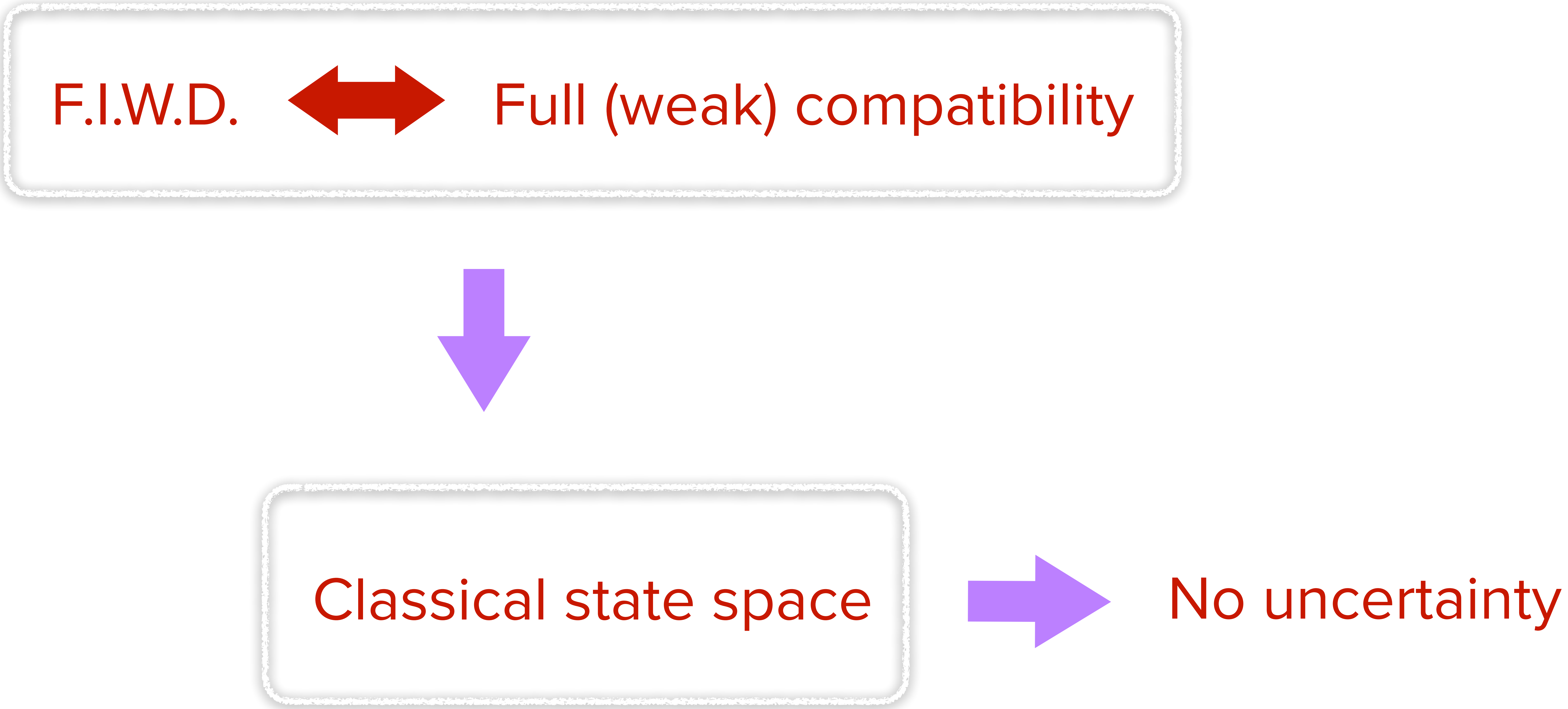

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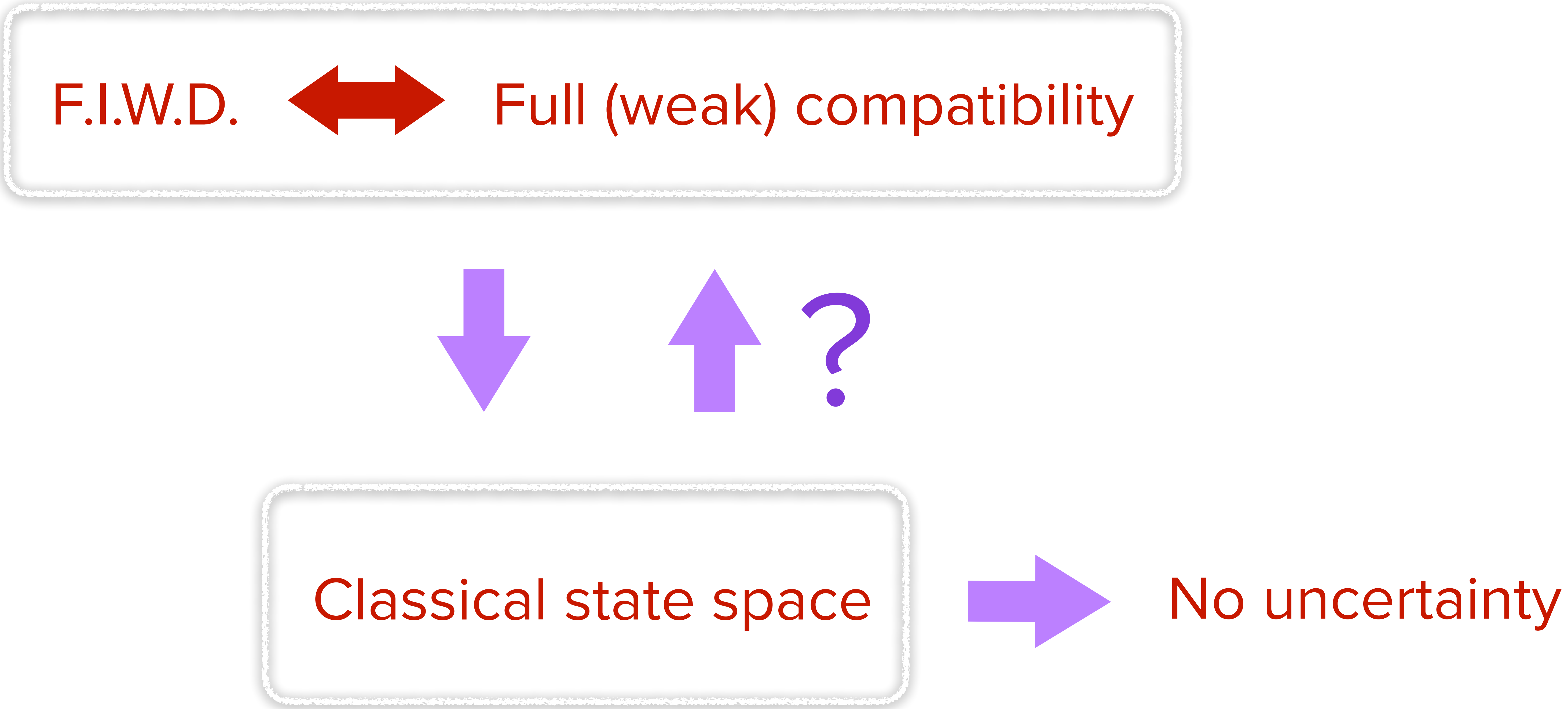

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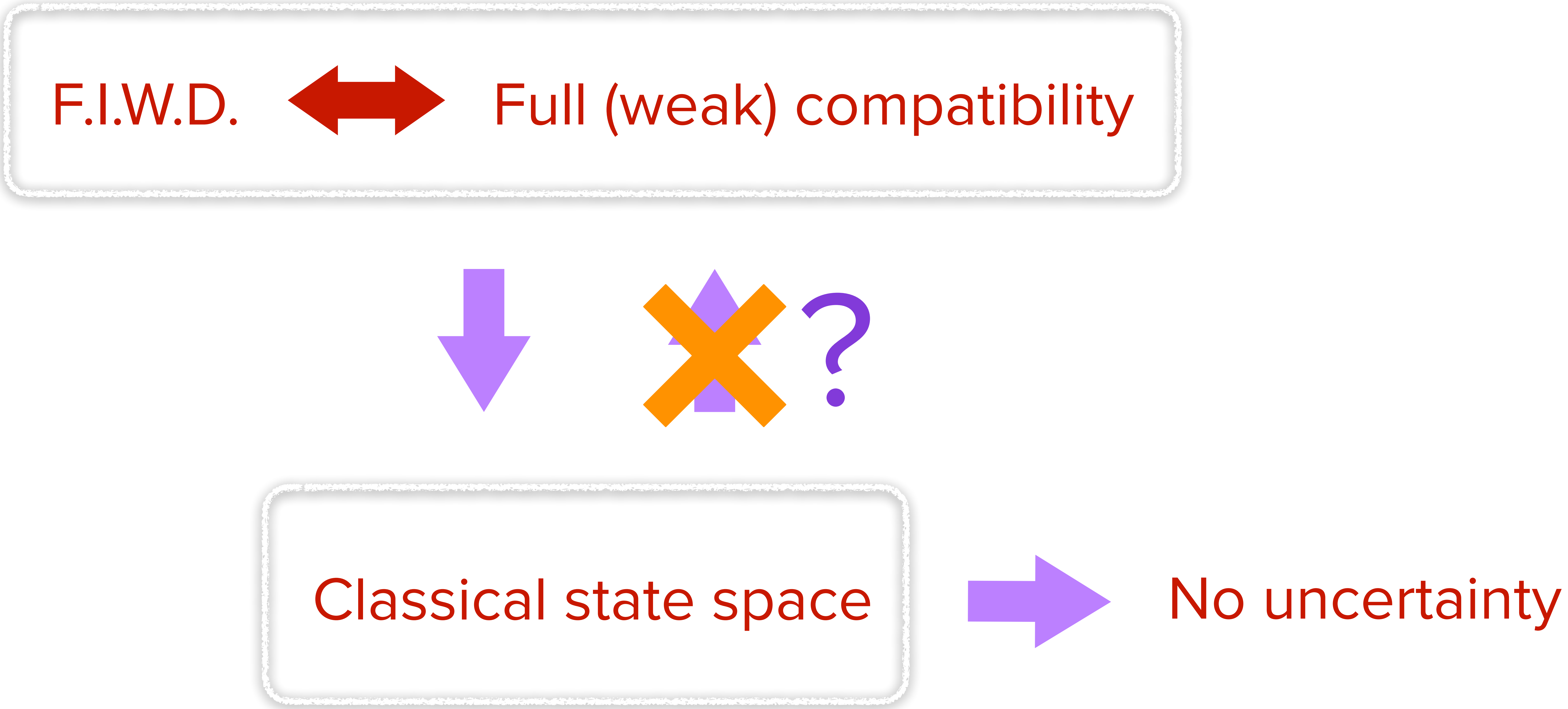

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MINIMAL CLASSICAL THEORY

- State and effect space: classical

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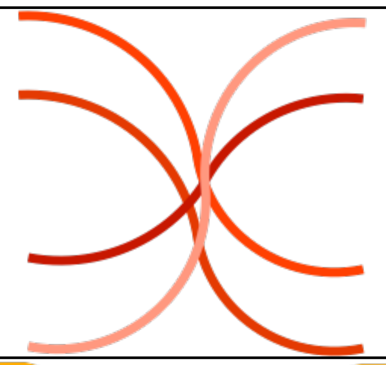
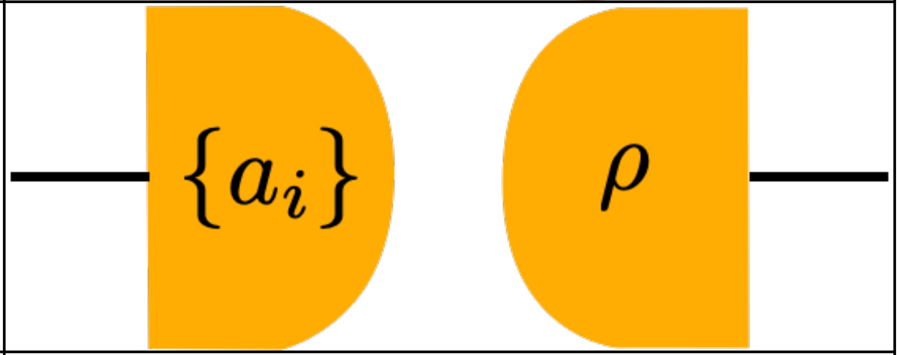
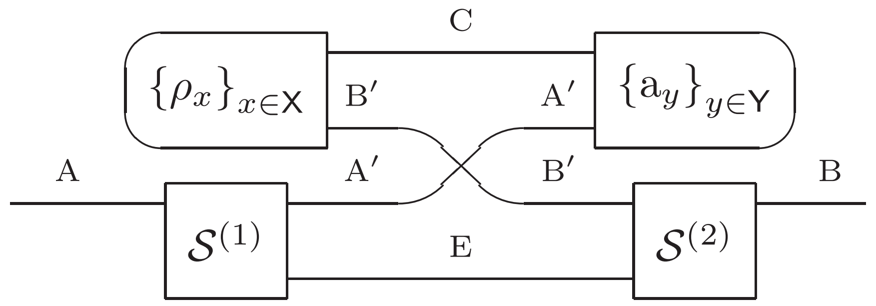
- Tests:

Permutations	
Measure-prepare (no conditioning)	
Compositions thereof	
Topological closure	?

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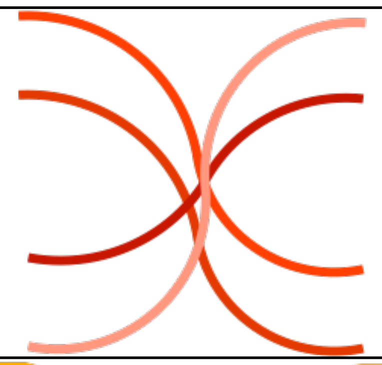
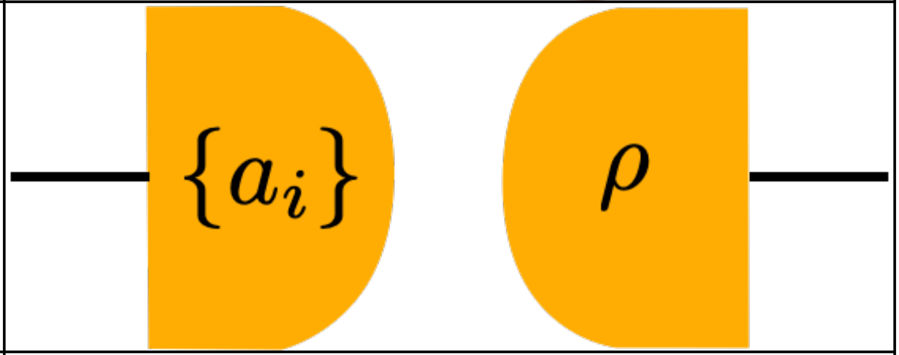
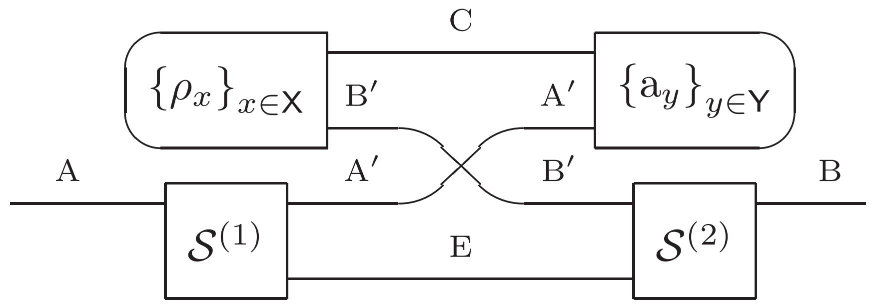
Permutations	
Measure-prepare (no conditioning)	
Compositions thereof	
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- Full characterisation of tests: yet unknown

M. Erba, PP, D. Rolino and A. Tosini, arXiv:2305.16931

MAIN PROPERTIES OF MCT

- In MCT all observation tests are compatible (trivial from CT) $\Rightarrow$ no uncertainty
- MCT has no-information without disturbance

M. Erba, PP, D. Rolino and A. Tosini, arXiv:2305.16931

SUMMARY

- Disturbance and correlations
- No information without disturbance = identity test is atomic
- Compatibility: strong and weak
  - Different kinds of irreversibility
- Full compatibility = full information without disturbance
  - Full compatibility implies classical state space
- MCT: Incompatibility does not necessarily imply uncertainty